A fully algebraic spectral coarse space for overlapping Schwarz methods

Alexander Heinlein

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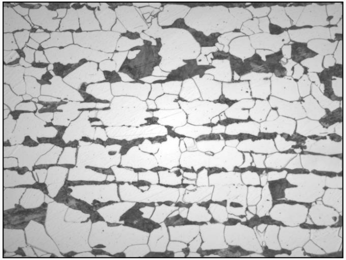
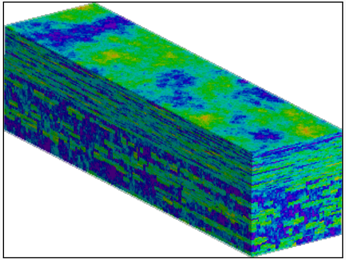
TU Delft

Based on joint work with Axel Klawonn, Jascha Knepper, Martin Lanser, Janine Weber (University of Cologne), Oliver Rheinbach (TU Bergakademie Freiberg), Kathrin Smetana (Stevens Institute of Technology), and Olof Widlund (New York University)

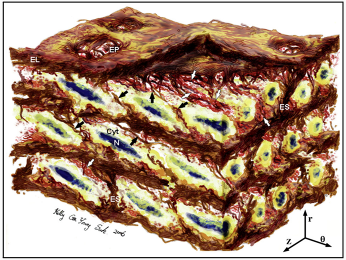
Highly heterogeneous multiscale problems appear in most areas of modern science and engineering, e.g., **composite materials**, **porous media**, and **turbulent transport in high Reynolds number flow**.



Microsection of a dual-phase steel. (Courtesy of Jörg Schröder, University of Duisburg-Essen, Germany; cooperation with ThyssenKrupp Steel.)



Groundwater flow: model 2 from the Tenth SPE Comparative Solution Project; cf. Christie and Blunt (2001).



Representation of the composition of a small segment of arterial walls; taken from O'Connell et al. (2008).

→ The solution of such problems requires a **high spatial and temporal resolution** but also poses **challenges to the solvers**.

Highly Heterogeneous Model Problem

Consider the **diffusion boundary value problem**: find u such that

$$
\begin{aligned}
-\nabla \cdot (\alpha(x) \nabla u(x)) &= f(x) \quad && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

with a **highly varying coefficient function** α. The corresponding weak formulation is: find $u \in H_0^1(\Omega)$, such that

$$
a_\Omega(u, v) = f(v) \qquad \forall v \in H_0^1(\Omega)
$$

with the bilinear form and linear functional

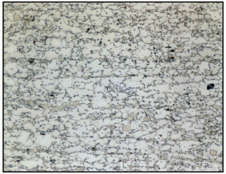
$$
a_\Omega(u, v) := \int_\Omega \alpha(x) (\nabla u(x))^T \nabla v(x) \, dx \text{ and } f(v) := \int_\Omega f(x) v(x) \, dx.
$$

Discretization using finite elements yields the linear system

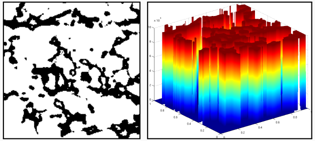
$$
\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}
$$

with stiffness matrix $\boldsymbol{A}$, discrete solution $\boldsymbol{u}$, and right hand side $\boldsymbol{f}$.

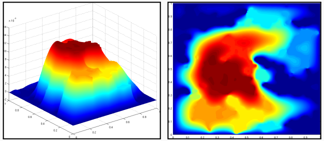
Original microsection of a dual-phase steel



Binary coefficient function

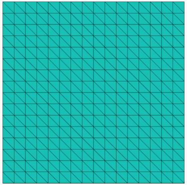


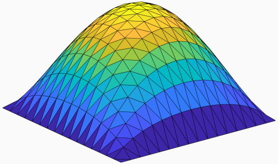
Solution of the BVP



Schwarz Domain Decomposition Preconditioners

Homogeneous Model Problem & Overlapping Domain Decomposition





Consider a **homogeneous diffusion model problem** $(\alpha(x) = 1)$:

$$-\Delta u = f \quad \text{in } \Omega = [0,1]^2,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

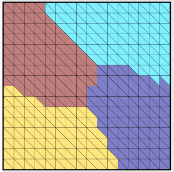
Discretization using finite elements yields the linear equation system

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}.$$

Overlapping Domain Decomposition

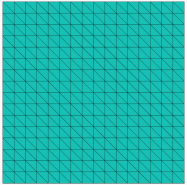
Overlapping Schwarz methods are based on **overlapping decompositions** of the computational domain Ω.

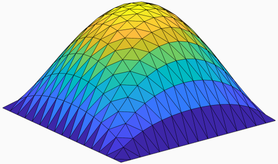
Overlapping subdomains $\Omega_1', \ldots, \Omega_N'$ can be constructed by **recursively adding layers of elements** to nonoverlapping subdomains $\Omega_1, \ldots, \Omega_N$.



Nonoverlap. DD

Homogeneous Model Problem & Overlapping Domain Decomposition





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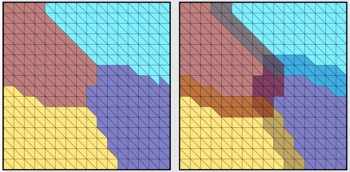
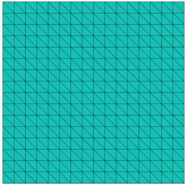
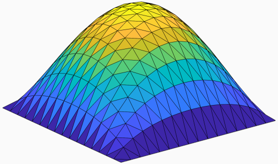


Nonoverlap. DD   Overlap $\delta = 1h$

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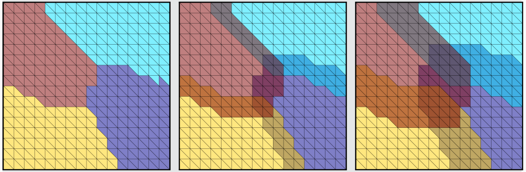
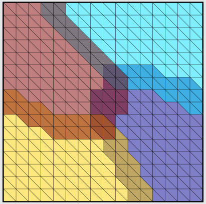


Nonoverlap. DD | Overlap $\delta = 1h$ | Overlap $\delta = 2h$

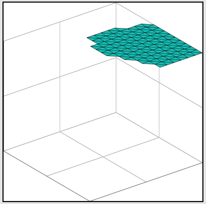
Two-Level Schwarz Preconditioners

One-Level Schwarz Preconditioner

Overlap $\delta = 1h$



Restriction $\boldsymbol{R}_i$ to Ω_i'



Based on an **overlapping domain decomposition**, we define a **one-level Schwarz operator**

$$\boldsymbol{M}_{\text{OS-1}}^{-1}\boldsymbol{A} = \sum_{i=1}^{N} \boldsymbol{R}_i^T \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \boldsymbol{A},$$

where $\boldsymbol{R}_i$ and $\boldsymbol{R}_i^T$ are restriction and prolongation operators corresponding to Ω_i', and $\boldsymbol{A}_i := \boldsymbol{R}_i \boldsymbol{A} \boldsymbol{R}_i^T$.

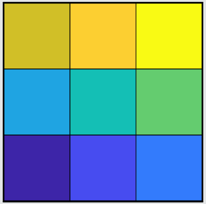
Condition number estimate:

$$\kappa\left(\boldsymbol{M}_{\text{OS-1}}^{-1}\boldsymbol{A}\right) \leq C\left(1 + \frac{1}{H\delta}\right)$$

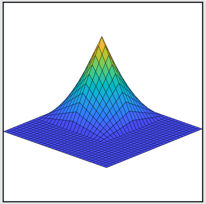
with subdomain size H and the width of the overlap δ.

Adding a Lagrangian Coarse Space

Coarse triangulation



Q1 basis function



The **two-level overlapping Schwarz operator** reads

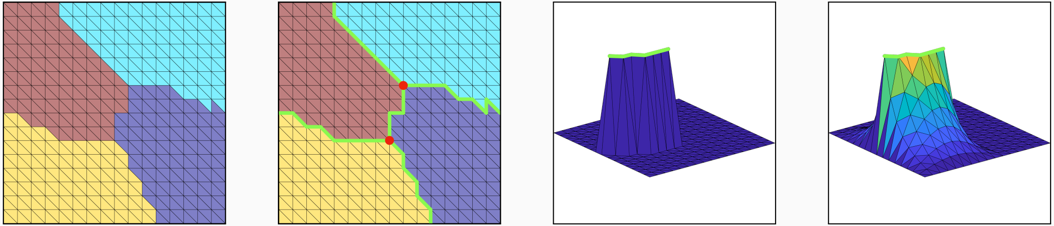
$$\boldsymbol{M}_{\text{OS-2}}^{-1}\boldsymbol{A} = \underbrace{\Phi \boldsymbol{A}_0^{-1} \Phi^T \boldsymbol{A}}_{\text{coarse level -- global}} + \underbrace{\sum_{i=1}^{N} \boldsymbol{R}_i^T \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \boldsymbol{A}}_{\text{first level -- local}},$$

where Φ contains the coarse basis functions and $\boldsymbol{A}_0 := \Phi^T \boldsymbol{A} \Phi$; cf., e.g., **Toselli, Widlund (2005)**.

A Lagrangian coarse basis requires a coarse triangulation (geometric information) $\rightarrow$ not algebraic

$$\Rightarrow \kappa\left(\boldsymbol{M}_{\text{OS-2}}\boldsymbol{A}\right) \leq C\left(1 + \frac{H}{\delta}\right)$$

Extension-Based GDSW Coarse Spaces



In **GDSW (Generalized–Dryja–Smith–Widlund) coarse spaces**, the coarse basis functions are chosen as **energy minimizing extensions** of functions Φ_Γ that are defined on the interface Γ:

$$\Phi = \begin{bmatrix} -A_{II}^{-1} A_{\Gamma I}^T \Phi_\Gamma \\ \Phi_\Gamma \end{bmatrix} = \begin{bmatrix} \Phi_I \\ \Phi_\Gamma \end{bmatrix}$$

The functions Φ_Γ are **restrictions of the null space of global Neumann matrix** to the **edges, vertices, and, in 3D, faces (partition of unity)** of the non-overlapping decomposition.

The **condition number of the GDSW operator** is bounded by

$$\kappa \left(M_{\mathrm{GDSW}}^{-1} A \right) \leq C \left(1 + \frac{H}{\delta} \right) \left(1 + \log \left(\frac{H}{h} \right) \right)^2 ;$$

cf. **Dohrmann, Klawonn, Widlund (2008), Dohrmann, Widlund (2009, 2010, 2012)**.

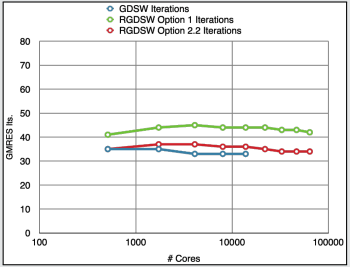
→ We only obtain the exponent 2 for very irregular subdomains.

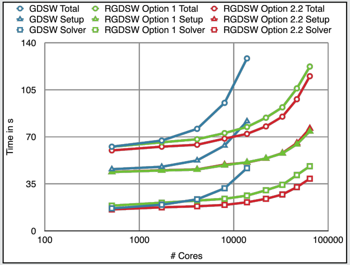
→ **Scalable and algebraic!**

Weak Scalability up to $64\,\mathrm{k}$ MPI Ranks / $1.7\,\mathrm{b}$ Unknowns (3D Poisson; Juqueen)

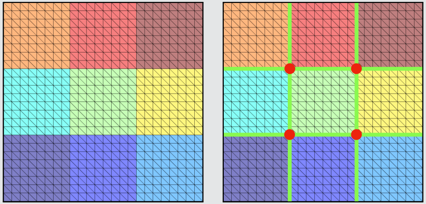
GDSW vs RGDSW (Reduced Dimension)

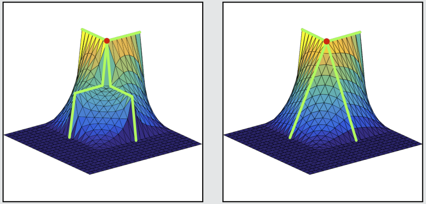
Heinlein, Klawonn, Rheinbach, Widlund (2019).





RGDSW (Reduced Dimension GDSW)





RGDSW option 1 RGDSW option 2.2

Reduced dimension GDSW coarse spaces are constructed from **nodal interface functions (different partition of unity)**; cf. Dohrmann, Widlund (2017).

FROSch (Fast and Robust Overlapping Schwarz) Framework in Trilinos

Software

- Object-oriented C++ domain decomposition solver framework with MPI-based distributed memory parallelization
- Part of Trilinos with support for both parallel linear algebra packages Epetra and Tpetra
- Node-level parallelization and performance portability on CPU and GPU architectures through Kokkos
- Accessible through unified Trilinos solver interface Stratimikos

Methoddology

- Parallel scalable multi-level Schwarz domain decomposition preconditioners
- Algebraic construction based on the parallel distributed system matrix
- Extension-based coarse spaces

Team (Active)

- Alexander Heinlein (TU Delft)
- Siva Rajamanickam (Sandia)
- Friederike Röver (TUBAF)
- Axel Klawonn (Uni Cologne)
- Oliver Rheinbach (TUBAF)
- Ichitaro Yamazaki (Sandia)

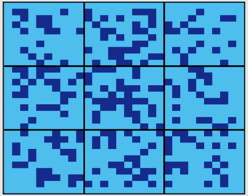
Observations for Heterogeneous Problems

Problem Configuration

Diffusion problem with **random binary coefficient** α: find u such that

$$-\nabla \cdot (\alpha(x)\nabla u(x)) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

Prec.	its.	κ
–	**>2 000**	$4.51 \cdot 10^8$
$M_{\text{OS-1}}^{-1}$	**>2 000**	$4.51 \cdot 10^8$
$M_{\text{OS-2}}^{-1}$	**586**	$5.56 \cdot 10^5$

Observations

→ For **heterogeneous coefficients**, the **condition number clearly deteriorates**. It depends on the **contrast of the coefficient function**

Heterogeneous Problem – Random Distribution

Problem Configuration

Diffusion problem with **random binary coefficient** α: find u such that

$$-\nabla \cdot (\alpha(x)\nabla u(x)) = f(x) \quad \text{in } \Omega,$$
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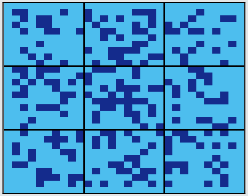
Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



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Observations

→ For **heterogeneous coefficients**, the **condition number clearly deteriorates**. It depends on the **contrast of the coefficient function**

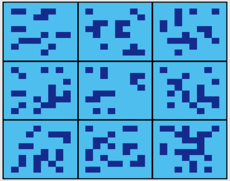
Let us consider some **pathological cases** to better understand the behavior of overlapping Schwarz methods for heterogeneous coefficient distributions.

Problem Configuration

Diffusion problem with **random binary coefficient α without high coefficients touching the interface**: find u such that

$$-\nabla \cdot (\alpha(x)\nabla u(x)) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

Prec.	its.	κ
–	**>2 000**	$7.99 \cdot 10^8$
$M_{\text{OS-1}}^{-1}$	**64**	133.16
$M_{\text{OS-2}}^{-1}$	**78**	139.15

Observations

→ In the first level, we **solve the subdomain problems exactly**

⇒ Jumps inside the subdomains are **not problematic**

→ Classical one- and two-level methods are **robust for jumps within the subdomains**

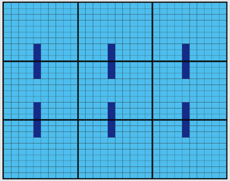
Heterogeneous Problem – Channels Across the Interface

Problem Configuration

Diffusion problem with **binary coefficient α with high contrast channels**: find u such that

$$-\nabla \cdot (\alpha(x)\nabla u(x)) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

Prec.	δ	its.	κ
–		**987**	$8.03 \cdot 10^8$
$M_{OS\text{-}1}^{-1}$	1h	**259**	$83.34 \cdot 10^6$
	2h	**216**	$5.56 \cdot 10^6$
	3h	**37**	91.97
$M_{OS\text{-}2}^{-1}$	1h	**163**	$4.70 \cdot 10^5$
	2h	**128**	$3.24 \cdot 10^5$
	3h	**44**	91.94

Observations

→ In case the **channels with high coefficient lie completely within the overlapping subdomains**, the method is again **robust**. Otherwise, the convergence **deteriorates**.

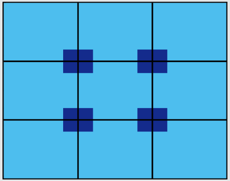
→ In general, it is **not practical to extend the overlap** until each high coefficient component lies **completely within one overlapping subdomain**.

Problem Configuration

Diffusion problem with **binary coefficient** α with **high coefficient inclusions at the vertices**: find u such that

$$-\nabla \cdot (\alpha(x)\nabla u(x)) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

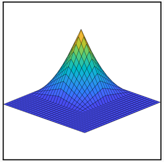
Prec.	its.	κ
–	**874**	$1.35 \cdot 10^9$
$M_{\text{OS-1}}^{-1}$	**163**	$4.06 \cdot 10^7$
$M_{\text{OS-2}}^{-1}$	**138**	$1.07 \cdot 10^6$
M_{MsFEM}^{-1}	**24**	8.05

Observations

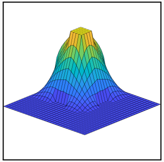
→ In general, one- or two-level Schwarz methods are **not robust** for **high coefficient inclusions at the vertices**

→ **Robustness can be retained** by using **multiscale finite element method (MsFEM)** type functions instead; cf. **Hou (1997), Efendiev and Hou (2009)**

Lagrangian function



MsFEM function



Heterogeneous Problem – Channels & Inclusions

Problem Configuration

Diffusion problem with **binary coefficient** α with **channels and vertex inclusions**: find u such that

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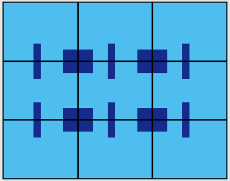
Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



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Prec.	its.	κ
–	**1708**	$1.16 \cdot 10^9$
$M_{\text{OS-1}}^{-1}$	**447**	$4.17 \cdot 10^7$
$M_{\text{OS-2}}^{-1}$	**268**	$1.10 \cdot 10^6$
M_{MsFEM}^{-1}	**117**	$4.34 \cdot 10^5$

Observations

→ **All of the aforementioned approaches fail** for this example.

→ Since we were able to **deal with the vertex inclusions**, the **problem has to be related to the edges**. How can we construct **suitable coarse basis functions** to deal with **coefficient jumps at the edges**?

Heterogeneous Problem – Channels & Inclusions

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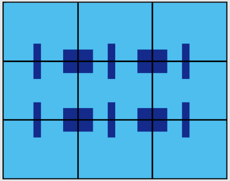

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Let us now discuss the **Schwarz theory** in order to construct a **robust coarse space for arbitrary heterogeneous problems**.

Influence of Heterogeneities on the Schwarz Theory

Schwarz Theory

In order to establish a condition number bound for $\kappa\left(\boldsymbol{M}_{\mathrm{ad}}^{-1}\boldsymbol{A}\right)$ based on the **abstract Schwarz framework**, we have to verify the following **three assumptions**:

Assumption 1: Stable Decomposition

There exists a constant C_0 such that, for every $u \in V$, there exists a decomposition $u = \sum_{i=0}^{N} \boldsymbol{R}_i^T u_i$, $u_i \in V_i$, with

$$\sum_{i=0}^{N} a_i(u_i, u_i) \leq C_0^2 a(u, u).$$

Assumption 2: Strengthened Cauchy-Schwarz Inequality

There exist constants $0 \leq \epsilon_{ij} \leq 1$, $1 \leq i, j \leq N$, such that

$$\left|a(\boldsymbol{R}_i^T u_i, \boldsymbol{R}_j^T u_j)\right| \leq \epsilon_{ij} \left(a(\boldsymbol{R}_i^T u_i, \boldsymbol{R}_i^T u_i)\right)^{1/2} \left(a(\boldsymbol{R}_j^T u_j, \boldsymbol{R}_j^T u_j)\right)^{1/2}$$

for $u_i \in V_i$ and $u_j \in V_j$. (Consider $\mathcal{E} = (\varepsilon_{ij})$ and $\rho(\mathcal{E})$ its spectral radius)

Assumption 3: Local Stability

There exists $\omega < 0$, such that

$$a(\boldsymbol{R}_i^T u_i, \boldsymbol{R}_i^T u_i) \leq \omega a_i(u_i, u_i), \quad u_i \in \operatorname{range}\left(\tilde{P}_i\right), \quad 0 \leq i \leq N.$$

Schwarz Theory

General Condition Number Bound

With Assumption 1–3, we have

$$\kappa\left(\boldsymbol{M}_{\mathrm{ad}}^{-1}\boldsymbol{A}\right) \leq C_0^2\omega\left(\rho\left(\mathcal{E}\right)+1\right)$$

for

$$\boldsymbol{M}_{\mathrm{ad}}\boldsymbol{A} = \sum\nolimits_{0=1}^{N} \boldsymbol{R}_i^T\boldsymbol{A}_i^{-1}\boldsymbol{R}_i\boldsymbol{A};$$

see, e.g., Toselli, Wildlund (2005).

To obtain a condition number bound for a specific additive Schwarz preconditioner, we have to estimate ω, $\rho\left(\mathcal{E}\right)$, and C_0^2.

The constants ω and $\rho\left(\mathcal{E}\right)$ can often easily be bounded.

Exact Solvers

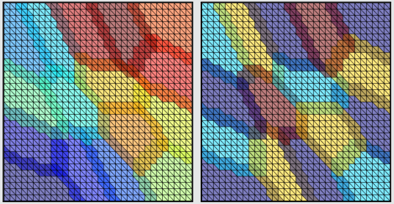
If we choose the local bilinear forms as

$$a_i(u_i,u_i) := a(\boldsymbol{R}_i^T u_i, \boldsymbol{R}_i^T u_i)$$

we obtain $\boldsymbol{A}_i = \boldsymbol{R}_i\boldsymbol{A}\boldsymbol{R}_i^T$ and $\omega = 1$.

$\rightarrow$ For exact **exact local and coarse solvers**, ω does not depend on the coefficient.

Coloring Constant



The spectral radius $\rho\left(\mathcal{E}\right)$ is bounded by the number of colors N^c of the domain decomposition.

$\rightarrow$ N^c depends only on the **domain decomposition** but not on the coefficient function.

In order to prove the existence of a stable decomposition

$$u = \sum_{i=0}^{N} R_i^T u_i \text{ and } \sum_{i=0}^{N} a_i(u_i, u_i) \leq C_0^2 a(u, u)$$

for a specific coarse space, the most essential estimate is

$$a_0(u_0, u_0) \leq C_0^2 a(u, u).$$

$\Rightarrow C_0^2$ will arise in the condition number estimate.

Homogeneous Diffusion

In the case of a diffusion problem with a constant coefficient,

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

this just corresponds to proving

$$|u_0|^2_{H^1(\Omega)} \leq C_0^2 \, |u|^2_{H^1(\Omega)} .$$

GDSW Coarse Space

In the proof for the GDSW preconditioner, we have

$$u_0(x) = \sum\nolimits_{\mathcal{V}} u(\mathcal{V})\theta_{\mathcal{V}}(x) + \sum\nolimits_{\mathcal{E}} \bar{u}_{\mathcal{E}}\theta_{\mathcal{E}}(x).$$

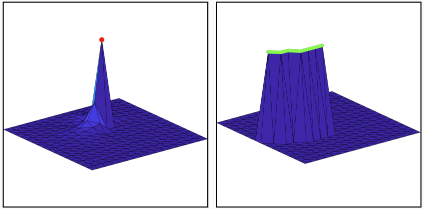
Then, using an inverse inequality for $\theta_{\mathcal{V}}$ and a discrete Sobolev inequality for $u - \bar{u}_{\Omega_i}$,

$$\left|(u(\mathcal{V}) - \bar{u}_{\Omega_i})\theta_{\mathcal{V}}\right|^2_{a,\Omega_i} \leq C(1 + \log(H/h)) \left\|u - \bar{u}_{\Omega_i}\right\|^2_{H^1(\Omega_i)}$$

and, similarly (estimating $\theta_{\mathcal{E}}$ adds another $(1 + \log(H/h))$),

$$\left|(\bar{u}_{\mathcal{E}} - \bar{u}_{\Omega_i})\theta_{\mathcal{E}}\right|^2_{a,\Omega_i} \leq C(1 + \log(H/h))^2 \left\|u - \bar{u}_{\Omega_i}\right\|^2_{H^1(\Omega_i)} .$$

Using a Poincaré inequality, we then obtain $|u|^2_{a,\Omega_i}$.



Discrete harmonic GDSW basis functions $\theta_{\mathcal{V}}$ and $\theta_{\mathcal{E}}$.

Stable Decomposition – GDSW Coarse Space

In order to prove the existence of a stable decomposition

$$u = \sum_{i=0}^{N} \boldsymbol{R}_i^T u_i \text{ and } \sum_{i=0}^{N} a_i(u_i, u_i) \leq C_0^2 a(u, u)$$

for a specific coarse space, the most essential estimate is

$$a_0(u_0, u_0) \leq C_0^2 a(u, u).$$

$\Rightarrow C_0^2$ will arise in the condition number estimate.

Heterogeneous Diffusion

In the case of a heterogeneous diffusion problem,

$$\begin{aligned} -\nabla \cdot (A(x) \cdot \nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

we have $a(u, v) = \int_\Omega A(x) \nabla u \cdot \nabla v \, dx$ and the **constants may depend on the contrast $\alpha_{\max}/\alpha_{\min}$.**

GDSW Coarse Space

In the proof for the GDSW preconditioner, we have

$$u_0(x) = \sum_{\mathcal{V}} u(\mathcal{V}) \theta_{\mathcal{V}}(x) + \sum_{\mathcal{E}} \bar{u}_{\mathcal{E}} \theta_{\mathcal{E}}(x).$$

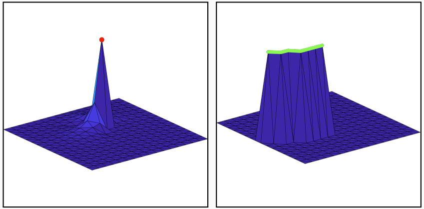
Then, using an inverse inequality for $\theta_{\mathcal{V}}$ and a discrete Sobolev inequality for $u - \bar{u}_{\Omega_i}$,

$$\left| (u(\mathcal{V}) - \bar{u}_{\Omega_i}) \theta_{\mathcal{V}} \right|_{a,\Omega_i}^2 \leq C(1 + \log(H/h)) \left\| u - \bar{u}_{\Omega_i} \right\|_{H^1(\Omega_i)}^2$$

and, similarly (estimating $\theta_{\mathcal{E}}$ adds another $(1 + \log(H/h))$),

$$\left| (\bar{u}_{\mathcal{E}} - \bar{u}_{\Omega_i}) \theta_{\mathcal{E}} \right|_{a,\Omega_i}^2 \leq C(1 + \log(H/h))^2 \left\| u - \bar{u}_{\Omega_i} \right\|_{H^1(\Omega_i)}^2.$$

Using a Poincaré inequality, we then obtain $|u|_{a,\Omega_i}^2$.



Discrete harmonic GDSW basis functions $\theta_{\mathcal{V}}$ and $\theta_{\mathcal{E}}$.

Stable Decomposition – Adaptive Coarse Spaces

In order to prove the existence of a stable decomposition

$$u = \sum_{i=0}^{N} \boldsymbol{R}_i^T u_i \text{ and } \sum_{i=0}^{N} a_i(u_i, u_i) \leq C_0^2 a(u, u)$$

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$$u = 0 \quad \text{on } \partial\Omega,$$

we have $a(u, v) = \int_\Omega A(x) \nabla u \cdot \nabla v \, dx$ and the **constants may depend on the contrast** $\alpha_{\max}/\alpha_{\min}$. $\Rightarrow$ **Remove dependence**

Idea of Adaptive Coarse Spaces

Ensure

$$a(u_0, u_0) \leq C_0^2 a(u, u)$$

by introducing two bilinear forms $c(\cdot, \cdot)$ and $d(\cdot, \cdot)$ with

$$a(u_0, u_0) \leq C_1 d(u_0, u_0) \quad \textbf{(high energy)}$$

and

$$c(u_0, u_0) \leq C_2 a(u, u), \quad \textbf{(low energy)}$$

where **C_1 and C_2 are independent of the coefficient function** and $u_0 := I_0 u$ is a suitable coarse function. Then, we **enhance the coarse space by all eigenvectors with eigenvalues below a tolerance** *tol* of the generalized eigenvalue problem

$$d(v, w) = \lambda \, c(v, w)$$

and obtain

$$a(u_0, u_0) \leq C_1 \, d(u_0, u_0) \leq C_1 \, tol \, c(u_0, u_0) \leq C_1 \, C_2 \, tol \, a(u, u)$$

without applying a Poincaré inequality. In practice, **eigenvalue problem is partitioned into many local eigenvalue problems** $\rightarrow$ parallelization!

Stable Decomposition – Adaptive Coarse Spaces

In order to prove the existence of a stable decomposition

$$u = \sum_{i=0}^{N} \mathbf{R}_i^T u_i \text{ and } \sum_{i=0}^{N} a_i(u_i, u_i) \leq C_0^2 a(u, u)$$

for a specific coarse space, the most essential estimate is

$$a_0(u_0, u_0) \leq C_0^2 a(u, u).$$

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Heterogeneous Diffusion

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we have $a(u, v) = \int_\Omega A(x) \nabla u \cdot \nabla v \, dx$ and the **constants may depend on the contrast** $\alpha_{\max}/\alpha_{\min}$.

Idea of Adaptive Coarse Spaces

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and

$$c(u_0, u_0) \leq C_2 a(u, u), \quad \textbf{(low energy)}$$

where **C_1 and C_2 are independent of the coefficient function** and $u_0 := I_0 u$ is a suitable coarse function. Then,

In practice, it is sufficient if C_1 and C_2 depend on either

- $\alpha_{\min}$ or
- $\alpha_{\max}$.

$\rightarrow$ In the **algebraic variant**, C_2 depends **only** on $\alpha_{\min}$.

$$a(u_0, u_0) \leq C_1 \, d(u_0, u_0) \leq C_1 \, tol \, c(u_0, u_0) \leq C_1 \, C_2 \, tol \, a(u, u)$$

without applying a Poincaré inequality. In practice, **eigenvalue problem is partitioned into many local eigenvalue problems** $\rightarrow$ parallelization!

Adaptive Coarse Spaces – OS-ACMS, AGDSW, and a Fully Algebraic Adaptive Coarse Space

Adaptive Coarse Spaces in Domain Decomposition Methods – Literature Overview

This list is **not exhaustive**:

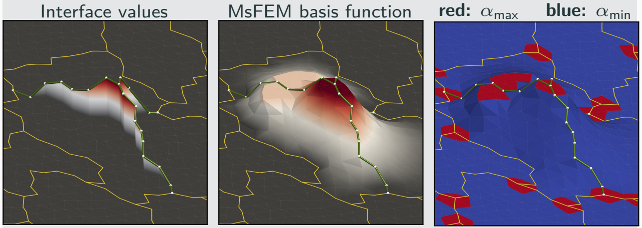
- **FETI & Neumann–Neumann:** Bjørstad and Krzyzanowski (2002); Bjørstad, Koster, and Krzyzanowski (2001); Rixen and Spillane (2013); Spillane (2015, 2016)
- **BDDC & FETI-DP:** Mandel and Sousedík (2007); Sousedík (2010); Sístek, Mandel, and Sousedík (2012); Dohrmann and Pechstein (2013, 2016); Klawonn, Radtke, and Rheinbach (2014, 2015, 2016); Klawonn, Kühn, and Rheinbach (2015, 2016, 2017); Kim and Chung (2015); Kim, Chung, and Wang (2017); Beirão da Veiga, Pavarino, Scacchi, Widlund, and Zampini (2017); Calvo and Widlund (2016); Oh, Widlund, Zampini, and Dohrmann (2017)
- **Overlapping Schwarz:** Galvis and Efendiev (2010, 2011); Nataf, Xiang, Dolean, and Spillane (2011); Spillane, Dolean, Hauret, Nataf, Pechstein, and Scheichl (2011); Gander, Loneland, and Rahman (preprint 2015); Eikeland, Marcinkowski, and Rahman (preprint 2016); Marcinkowski and Rahman (2018); Al Daas, Grigori, Jolivet, Tournier (2021); Bastian, Scheichl, Seelinger, and Strehlow (2022); Spillane (preprint 2021, preprint 2021); Bootland, Dolean, Graham, Ma, Scheichl (preprint 2021)
- Approaches for overlapping Schwarz methods in **this talk**:
  - **OS-ACMS:** Heinlein, Klawonn, Knepper, Rheinbach (2018)
  - **AGDSW:** Heinlein, Klawonn, Knepper, Rheinbach (2019, 2019), Heinlein, Klawonn, Knepper, Rheinbach Widlund (2022)
  - **Fully Algebraic Coarse Space:** Heinlein and Smetana (Preprint: arXiv:2207.05559)

There is also related work on multigrid methods, such as **AMGe** by Brezina, Cleary, Falgout, Henson, Jones, Manteuffel, McCormick, Ruge (2000).

OS-ACMS – An Adaptive Coarse Space Based on the ACMS Discretization

As in the ACMS finite element space, we construct a coarse space composed of **MsFEM-type nodal and coupling basis functions**. However, in order to obtain a robust coarse space, the construction has to be slightly **modified**; see **Heinlein, Klawonn, Knepper, Rheinbach (2018)**.

MsFEM Type Basis Functions



We define the interface values as follows:

$$\varphi_v(v') = \delta_{v,v'} \qquad \forall v' \in \mathcal{V} \qquad \textit{(Kronecker property)}$$

$$\varphi_v|_e = E_{\partial e \to \Omega_e}\left(\varphi_v|_{\partial e}\right) \qquad \forall e \in \mathcal{E} \qquad \textit{(Energy min. ext.)}$$

The interior values are then obtained by an energy minimizing extension into the interior:

$$\varphi_v = E_{\Gamma \to \Omega}\left(\varphi_v|_\Gamma\right)$$

Energy Minimizing Extensions

The energy minimizing extension $E_{\partial\Omega \to \Omega}(v)$ of the function v defined on $\partial\Omega$ is given by the solution of the boundary value problem:

$$\begin{aligned} a_\Omega\left(E_{\partial\Omega\to\Omega}(v), w\right) &= 0 \quad \forall w \in V_\Omega^0, \\ E_{\partial\Omega\to\Omega}(v) &= v \quad \text{on } \partial\Omega. \end{aligned}$$

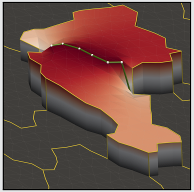
This is equivalent to solving

$$E_{\partial\Omega\to\Omega}(v) = \arg \min_{\substack{w \in V_\Omega \\ w_{\partial\Omega} = v}} a_\Omega(w, w).$$

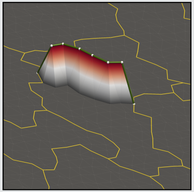
OS-ACMS – An Adaptive Coarse Space Based on the ACMS Discretization

OS-ACMS Edge Basis Functions

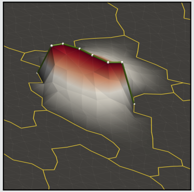
Low energy extension $E_{\overline{e}\to\Omega_e}(\cdot)$



High energy extension $R_{e\to\Omega_e}(\cdot)$



Ext. into the interior



First, we solve the following eigenvalue problem for each edge $e \in \mathcal{E}$:

$$a_{\Omega_e}\left(E_{\overline{e}\to\Omega_e}(\tau_{e,*}), E_{\overline{e}\to\Omega_e}(\theta)\right) = \lambda_{e,*}\, a_{\Omega_e}\left(R_{e\to\Omega_e}(\tau_{e,*}), R_{e\to\Omega_e}(\theta)\right) \quad \forall \theta \in V_e^0$$

Then, we select all eigenfunctions $\tau_{e,*}$ with $\lambda_{e,*}$ below a **user-chosen threshold** TOL. We then extend $\tau_{e,*}$ by zero onto Γ and with minimum energy into Ω to obtain the corresponding basis functions:

$$\varphi_{e,*} = E_{\Gamma\to\Omega}\left(R_{e\to\Gamma}(\tau_{e,*})\right)$$

Condition Number Bound

Using the coarse space $V_{\text{OS-ACMS}} = \{\varphi_v\} \cup \{\varphi_e\}$ in the two-level Schwarz preconditioner, we obtain

$$\kappa\left(\boldsymbol{M}_{\text{OS-ACMS}}^{-1}\boldsymbol{A}\right) \leq C\,(1/TOL),$$

where C is independent of H, h, and the contrast of the coefficient function α.

AGDSW – An Adaptive GDSW Coarse Space

The **adaptive GDSW (AGDSW) coarse space** is a related approach, which also depends on a **partition of the domain decomposition interface** into edges and vertices. It differs from OS-ACMS as follows:

- Instead of MsFEM functions, we use the **much simpler and cheaper** **GDSW vertex basis functions**
- The edge eigenvalue problem has to be **modified accordingly**

As a result, the AGDSW coarse space

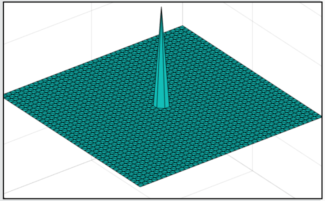
- always **contains the classical GDSW coarse space**.

Cf. Heinlein, Klawonn, Knepper, Rheinbach (2019, 2019, 2022).

AGDSW Vertex Basis Function

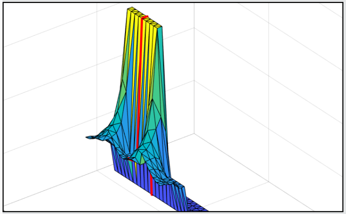
The interior values are then obtained by extending 1 be zero onto the remainder of the interface followed by an energy minimizing extension into the interior:

$$\varphi_v = E_{\Gamma \to \Omega}\left(R_{v \to \Gamma}\left(\mathbb{1}_v\right)\right)$$

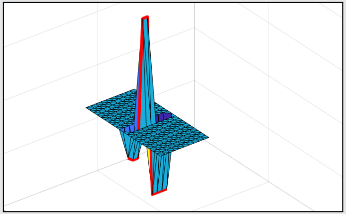


AGDSW Edge Basis Functions

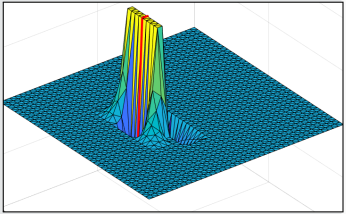
Low energy extension $E_{e \to \Omega_e}(\cdot)$



High energy extension $R_{e \to \Omega_e}(\cdot)$



Ext. into the interior



First, we solve the following eigenvalue problem for each edge $e \in \mathcal{E}$:

$$a_{\Omega_e}\left(E_{e \to \Omega_e}(\tau_{e,*}), E_{e \to \Omega_e}(\theta)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e \to \Omega_e}(\tau_{e,*}), R_{e \to \Omega_e}(\theta)\right) \quad \forall \theta \in V_e$$

Then, we select eigenfunctions using the threshold TOL and extend the edge values to Ω:

$$\varphi_{e,*} = E_{\Gamma \to \Omega}\left(R_{e \to \Gamma}(\tau_{e,*})\right)$$

Condition Number Bound

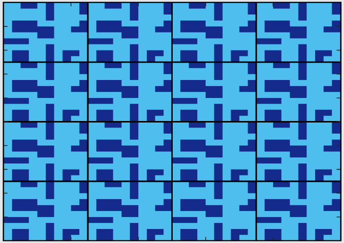
Using the coarse space $V_{\text{OS-ACMS}} = \{\varphi_v\} \cup \{\varphi_e\}$ in the two-level Schwarz preconditioner, we obtain

$$\kappa\left(\boldsymbol{M}_{\text{AGDSW}}^{-1}\boldsymbol{A}\right) \leq C\,(1/TOL),$$

where C is independent of H, h, and the contrast of the coefficient function α.

Numerical Results of Adaptive Coarse Spaces (2D)

Example 1

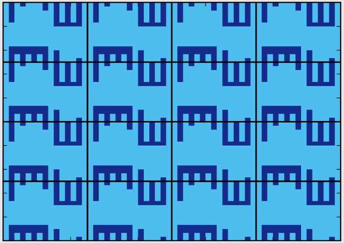


dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

4×4 subdomains, $H/h = 30$, $\delta = 2h$

V_0	*tol*	it.	κ	dim V_0
V_{MsFEM}	-	**199**	$7.8 \cdot 10^5$	**9**
$V_{\text{OS-ACMS}}$	10^{-2}	**23**	5.1	**69**
V_{SHEM}	10^{-3}	**20**	4.3	**69**
V_{AGDSW}	10^{-2}	**29**	7.2	**93**

Example 2



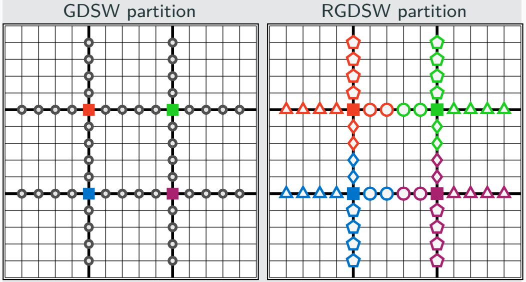
dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

4×4 subdomains, $H/h = 30$, $\delta = 2h$

V_0	*tol*	it.	κ	dim V_0
V_{MsFEM}	-	**282**	$3.8 \cdot 10^7$	9
$V_{\text{OS-ACMS}}$	10^{-2}	**41**	13.2	**33**
V_{SHEM}	10^{-3}	**29**	6.4	**93**
V_{AGDSW}	10^{-2}	**42**	16.5	**45**

SHEM by Gander, Loneland, Rahman (TR 2015), **OS-ACMS** from H., Klawonn, Knepper, Rheinbach (2018), **AGDSW** from H., Klawonn, Knepper, Rheinbach (2019)

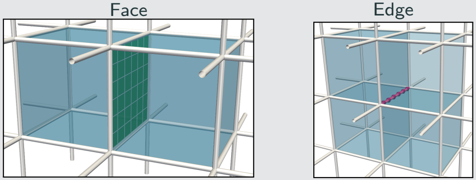
Reducing the Coarse Space Dimension



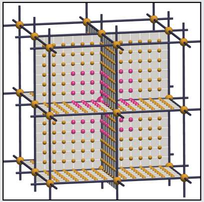
As in the reduced dimension GDSW (RGDSW) approach, we partition the interface into **interface components centered around the vertices**. On these interface components, we solve (slightly modified) eigenvalue problems.

Cf. Heinlein, Klawonn, Knepper, Rheinbach (2021) and Heinlein, Klawonn, Knepper, Rheinbach, Widlund (2022).

Extension to Three Dimensions

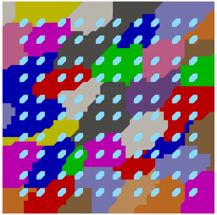


- In AGDSW, we have to solve **face and edge eigenvalue problems**
- In RAGDSW, only the **interface components change**

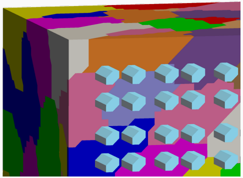


RGDSW interface component

cross section



detailed view of partially peeled beams



V_0	*tol*	iter	κ	dim V_0	$\frac{\dim V_0}{\dim V^h}$
GDSW	–	>2 000	$3.1 \cdot 10^5$	9 996	2.51%
RGDSW	–	>2 000	$3.9 \cdot 10^5$	3 358	0.84%
AGDSW	0.100	71	41.1	14 439	3.63%
AGDSW	0.050	90	59.5	13 945	3.50%
AGDSW	0.010	132	161.1	13 763	3.46%
RAGDSW	0.100	67	34.6	8 249	2.07%
RAGDSW	0.050	88	61.3	7 683	1.93%
RAGDSW	0.010	114	117.4	7 501	1.88%

Heterogeneous linear elasticity problem

- Ω: cube; Dirichlet boundary condition on $\partial\Omega$.
- Structured tetrahedral mesh; 132 651 nodes (397 953 DOFs); unstructured domain decomposition (METIS); 125 subdomains.
- Poisson ration $\nu = 0.4$.
- Young modulus: elements with $E(T) = 10^6$ in light blue (beams); remainder set to $E(T) = 1$.
- Right hand side $f \equiv 1$.
- Overlap: two layers of finite elements.

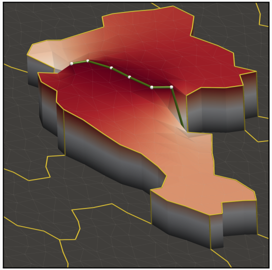
- RAGDSW: 45% reduction of coarse space dimension compared to AGDSW (highlighted line).
- RAGDSW: smaller coarse space dimension compared to GDSW and even robust!

Neumann Matrices and Algebraicity

The **low energy property**

$$c(u_0, u_0) \leq C_2 a(u, u)$$

of the **left hand side in the eigenvalue problems** of the OS-ACMS and AGDSW methods is satisfied due to the use of **Neumann boundary conditions**:



$$a_{\Omega_e}\left(E_{\overline{e}\to\Omega_e}(\tau_{e,*}), E_{\overline{e}\to\Omega_e}(\theta)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}(\tau_{e,*}), R_{e\to\Omega_e}(\theta)\right) \quad \forall \theta \in V_e^0 \qquad \text{(OS-ACMS)}$$

$$a_{\Omega_e}\left(E_{e\to\Omega_e}(\tau_{e,*}), E_{e\to\Omega_e}(\theta)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}(\tau_{e,*}), R_{e\to\Omega_e}(\theta)\right) \quad \forall \theta \in V_e^0 \qquad \text{(AGDSW)}$$

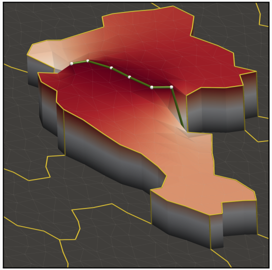
In both approaches, the right hand side matrix just corresponds to the submatrix $\boldsymbol{A}_{ee}$ of $\boldsymbol{A}$ corresponding to the edge e, whereas the Neumann matrices on the left hand sides cannot be extracted from the fully assembled matrix $\boldsymbol{A}$.

Neumann Matrices and Algebraicity

The **low energy property**

$$c(u_0, u_0) \leq C_2 a(u, u)$$

of the **left hand side in the eigenvalue problems** of the OS-ACMS and AGDSW methods is satisfied due to the use of **Neumann boundary conditions**:

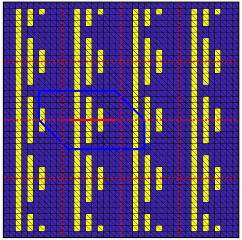


$$\begin{aligned} a_{\Omega_e}\left(E_{\overline{e}\to\Omega_e}(\tau_{e,*}), E_{\overline{e}\to\Omega_e}(\theta)\right) &= \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}(\tau_{e,*}), R_{e\to\Omega_e}(\theta)\right) \quad \forall \theta \in V_e^0 \qquad \text{(OS-ACMS)} \\ a_{\Omega_e}\left(E_{e\to\Omega_e}(\tau_{e,*}), E_{e\to\Omega_e}(\theta)\right) &= \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}(\tau_{e,*}), R_{e\to\Omega_e}(\theta)\right) \quad \forall \theta \in V_e^0 \qquad \text{(AGDSW)} \end{aligned}$$

In both approaches, the right hand side matrix just corresponds to the submatrix $\boldsymbol{A}_{ee}$ of $\boldsymbol{A}$ corresponding to the edge e, whereas the Neumann matrices on the left hand sides cannot be extracted from the fully assembled matrix $\boldsymbol{A}$.

$\rightarrow$ Both approaches are **not algebraic**

Fully Algebraic Adaptive Coarse Space



We can make use of the a-orthogonal decomposition

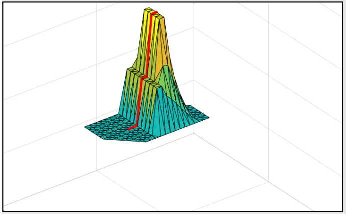
$$V_{\Omega_e} = V^0_{\Omega_e} \oplus \underbrace{\left\{ E_{\partial\Omega_e \to \Omega_e}(v) : v \in V_{\partial\Omega_e} \right\}}_{=: V_{\Omega_e,\text{harm}}}$$

to ***"split the AGDSW eigenvalue problem"*** into two:

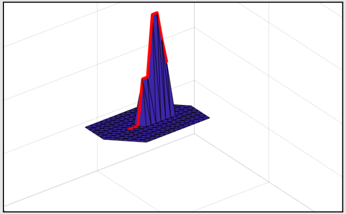
- **Dirichlet eigenvalue problem** on $V^0_{\Omega_e}$
- **Transfer eigenvalue problem** on $V_{\Omega_e,\text{harm}}$; cf. **Smetana, Patera (2016)**

Dirichlet Eigenvalue Problem

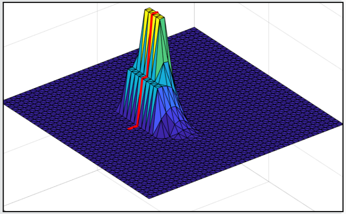
Low energy ext. (lhs evp)



High energy ext. (rhs evp)



Basis function



We solve the eigenvalue problem, choose $\lambda_{e,*} < TOL_1$, and extend the basis functions to Ω as before:

$$a_{\Omega_e}\left(E^{\partial\Omega_e}_{e\to\Omega_e}(\tau_{e,*}), E^{\partial\Omega_e}_{e\to\Omega_e}(\theta)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}(\tau_{e,*}), R_{e\to\Omega_e}(\theta)\right) \quad \forall \theta \in V^0_e$$

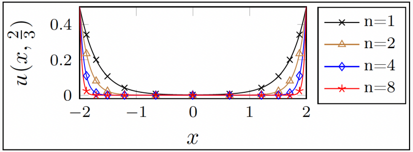
Solution Space of Elliptic PDEs is Locally Low-Dimensional

- Consider $\omega^{out} = (-2,2) \times (0,1)$
  $-\Delta u = 0$ in ω^{out},
  $u_y(x,1) = u_y(x,0) = 0.$
- plus arbitrary Dirichlet b.c. on $\partial\omega^{out}$.

- separation of variables: all local solutions on ω^{out} have the form

$$u(x,y) = a_0 + b_0 x + \sum_{n=1}^{\infty} \cos(n\pi y)[a_n \cosh(n\pi x) + b_n \sinh(n\pi x)]$$

- Solution $u(x, \frac{2}{3})$ for boundary cond. $-\cos(n\pi y)$ at $x=-2, x=2$:



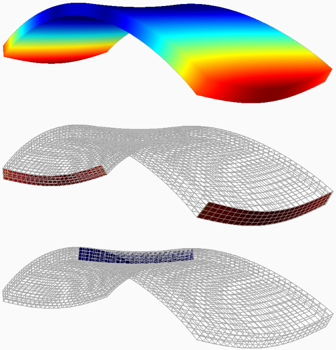
A very low-dimensional subspace on ω^{in} will already yield a very good approximation

Cf. Smetana, Patera (2016)

Constructing Local Reduced Spaces via a Transfer Operator

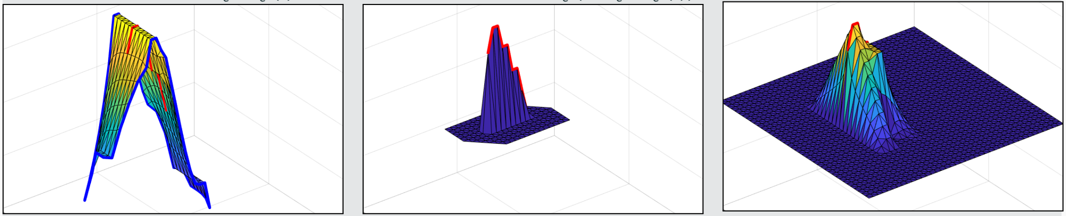
Introduce transfer operator $\mathcal{T}$:

- ... acts on the space of local solutions of the PDE and maps values ζ on $\partial\omega^{out}$ to ω^{in}
- ... by solving the PDE locally with Dirichlet boundary values ζ
- ... and restricting the local solution to ω^{in}



Cf. **Smetana, Patera (2016)**

Transfer Eigenvalue Problem



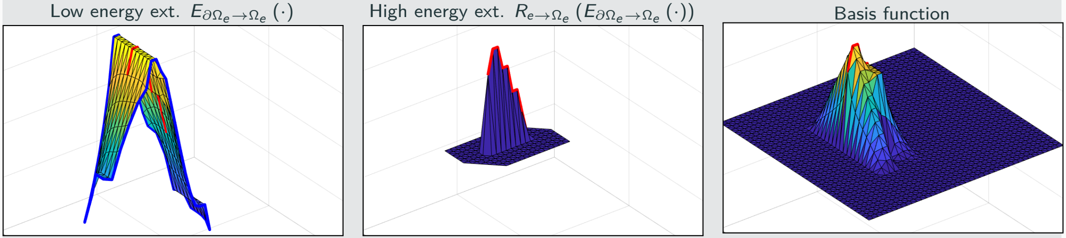
The transfer eigenvalue problem is based on **Smetana, Patera (2016)**. Different from all the eigenvalue problems before, it is solved on the boundary of Ω_e:

$$a_{\Omega_e}\left(E_{\partial\Omega_e\to\Omega_e}(\eta_{e,*}), E_{\partial\Omega_e\to\Omega_e}(\theta)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}(E_{\partial\Omega_e\to\Omega_e}(\tau_{e,*})), R_{e\to\Omega_e}(E_{\partial\Omega_e\to\Omega_e}(\theta))\right) \quad \forall \theta \in V^0_{\partial\Omega_e}$$

We select all eigenfunctions $\eta_{e,*}$ with $\lambda_{e,*}$ above a second **user-chosen threshold** TOL_2. Then, we first compute the edge values $\tau_{e,*} = E_{\partial\Omega_e\to\Omega_e}(\eta_{e,*})|_e$ and then extend them into the interior

$$\varphi_{e,*} = E_{\Gamma\to\Omega}(R_{e\to\Gamma}(\tau_{e,*}))$$

Transfer Eigenvalue Problem



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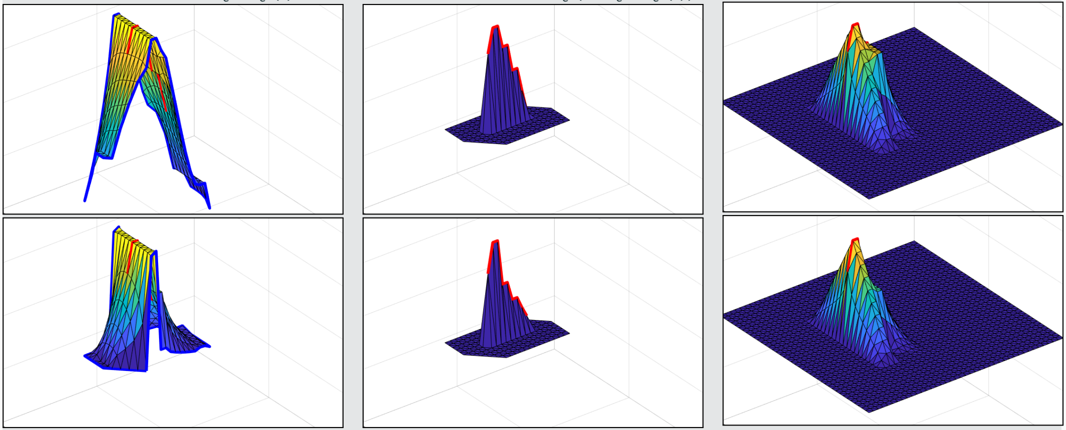
$$\varphi_{e,*} = E_{\Gamma \to \Omega}(R_{e \to \Gamma}(\tau_{e,*}))$$

→ Even though **no Neumann matrices are needed to compute** $E_{\partial\Omega_e \to \Omega_e}(\theta)$, **Neumann matrices are needed to evaluate** $a_{\Omega_e}(\cdot,\cdot)$ for functions with nonnegative trace on $\partial\Omega_e$

Algebraic Transfer Eigenvalue Problem

Low energy ext. $E_{\partial\Omega_e \to \Omega_e}(\cdot)$ | High energy ext. $R_{e \to \Omega_e}(E_{\partial\Omega_e \to \Omega_e}(\cdot))$ | Basis function for $a_{\Omega_e}(\cdot, \cdot)$



Low energy ext. $E_{\partial\Omega_e \to \Omega_e}(\cdot)$ | High energy ext. $R_{e \to \Omega_e}(E_{\partial\Omega_e \to \Omega_e}(\cdot))$ | Basis function for $(\cdot, \cdot)_{l_2(\partial\Omega_e)}$

In order to obtain an algebraic transfer eigenvalue problem, we replace $a_{\Omega_e}(\cdot, \cdot)$ by $(\cdot, \cdot)_{l_2(\partial\Omega_e)}$:

$$\left(E_{\partial\Omega_e \to \Omega_e}(\tau_{e,*}), E_{\partial\Omega_e \to \Omega_e}(\theta)\right)_{l_2(\partial\Omega_e)} = \lambda_{e,*} a_{\Omega_e}\left(R_{e \to \Omega_e}(E_{\partial\Omega_e \to \Omega_e}(\tau_{e,*})), R_{e \to \Omega_e}(E_{\partial\Omega_e \to \Omega_e}(\theta))\right) \quad \forall \theta \in V^0_{\partial\Omega_e}$$

Fully Algebraic Adaptive Coarse Space – Condition Number Bound

Condition Number Estimate (Non-Algebraic Variant)

Using the non-algebraic eigenvalue problem (transfer eigenvalue problem with $a_{\Omega_e}(\cdot,\cdot)$), we obtain a condition number of the form:

$$\kappa\left(\boldsymbol{M}_{\text{DIR\&TR}}^{-1}\boldsymbol{A}\right) \leq C \max\left(\frac{1}{TOL_1}, TOL_2\right),$$

where C is independent of H, h, and the contrast of the coefficient function α.

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$$\kappa\left(\boldsymbol{M}_{\text{DIR\&TR}}^{-1}\boldsymbol{A}\right) \leq C \max\left\{\frac{1}{TOL_1}, \frac{TOL_2}{\alpha_{\min}}\right\},$$

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Cf. **Heinlein and Smetana (Preprint: arXiv:2207.05559)**.

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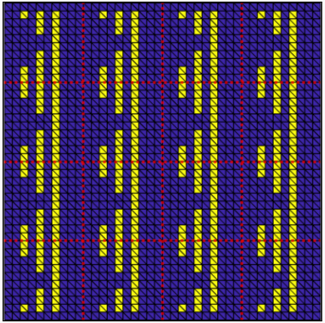
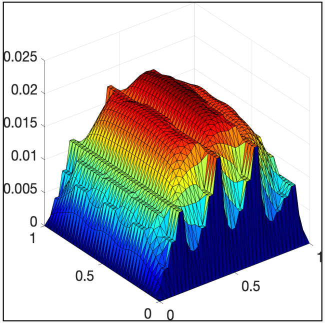
where C is independent of H, h, and the contrast of the coefficient function α.

$\rightarrow$ The $\alpha_{\min}$ arises from the fact that

$$\alpha_{min}\|\theta\|^2_{l_2(\partial\Omega_e)} \leq C\|E_{\partial\Omega_e\rightarrow\Omega_e}(\theta)\|^2_{a,\Omega_e} \quad \forall\theta\in V_{\partial\Omega_e}.$$

Cf. **Heinlein and Smetana (Preprint: arXiv:2207.05559)**.



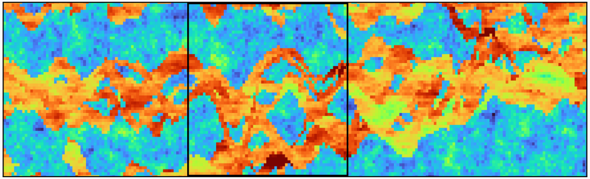


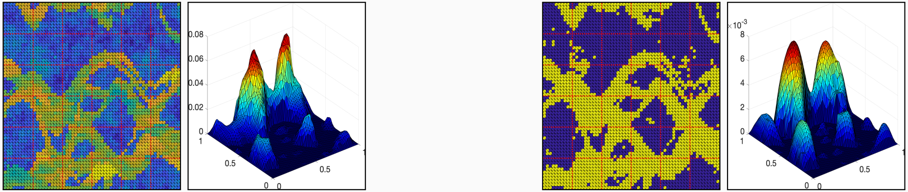
yellow: $\alpha = 10^6$ **blue:** $\alpha = 1$

V_0	variant	TOL_{DIR}	TOL_{TR}	TOL_{POD}	dim V_0	κ	# its.
V_{GDSW}	-	-	-	-	33	$2.7 \cdot 10^5$	118
V_{AGDSW}	-	$1.0 \cdot 10^{-2}$			57	7.4	24
$V_{\text{DIR\&TR}}$	$a_{\Omega_e}(\cdot,\cdot)$	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^{1}$	$1.0 \cdot 10^{-5}$	57	7.2	24
$V_{\text{DIR\&TR}}$	$(\cdot,\cdot)_{l_2(\partial\Omega_e)}$	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^{1}$	$1.0 \cdot 10^{-5}$	57	7.2	24

→ In order to get rid of potential **linear dependencies** between the V_{DIR} and V_{TR} spaces, apply a **proper orthogonal decomposition (POD)** with threshold TOL_{POD} for each edge.

Layer 70 from model 2 of the SPE10 benchmark; cf. Christie and Blunt (2001)





V_0	variant	TOL_{DIR}	TOL_{TR}	TOL_{POD}	dim V_0	κ	# its.
V_{GDSW}	-	-	-	-	85	$2.0 \cdot 10^5$	57
V_{AGDSW}	-		$1.0 \cdot 10^{-2}$		93	19.3	38
$V_{\text{DIR\&TR}}$	$a_{\Omega_e}(\cdot,\cdot)$	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^5$	$1.0 \cdot 10^{-5}$	90	19.4	39
$V_{\text{DIR\&TR}}$	$(\cdot,\cdot)_{l_2(\partial\Omega_e)}$	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^5$	$1.0 \cdot 10^{-5}$	147	9.6	31
Original coefficient $\alpha_{\max} \approx 10^4, \alpha_{\min} \approx 10^{-2}$ (without thresholding)							
V_{GDSW}	-	-	-	-	85	20.6	42

Thank you for your attention!

Summary

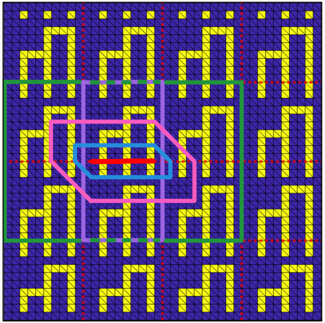
- Using adaptive coarse spaces we are able to **retain robustness** of two-level Schwarz preconditioners for **highly heterogeneous problems**:
  - The **support and computation** of the coarse basis functions are **local**, however, the computation comes at **substantial computational cost**.
  - The condition number bound is **independent of the contrast of the coefficient function**.
- The algebraic variant requires the solution of **two eigenvalue problems**. The **minimum value of the coefficient function** appears in the condition number bound.

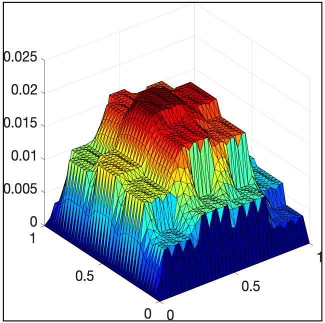
Outlook

- Efficient solution of the local eigenvalue problems, for instance, using inexact eigensolvers
- Parallel implementation of adaptive coarse spaces

Additional Results

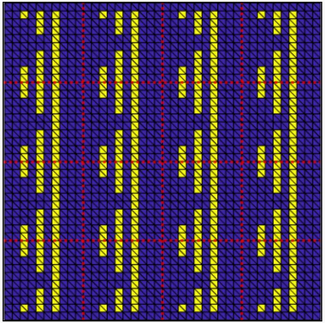
Numerical Results – Comb Type Coefficient Function

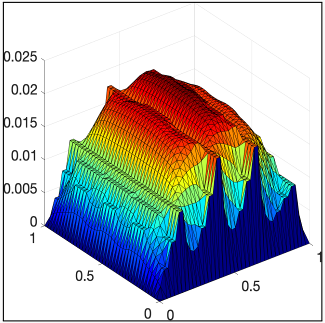




yellow: $\alpha = 10^6$ **blue:** $\alpha = 1$

V_0	Ω_e	TOL_{DIR}	TOL_{TR}	TOL_{POD}	dim V_0	κ	# its.
V_{AGDSW}	Ω_e^{2h}	10^{-3}	–	–	57	7.1	24
	Ω_e^{5h}	10^{-3}	–	–	45	12.6	26
	Ω_e^{H}	10^{-3}	–	–	33	24.1	31
	–	10^{-3}	–	–	33	24.1	31
$V_{\text{DIR\&TR}}$	Ω_e^{2h}	10^{-3}	$\mathbf{10^6}$	10^{-5}	57	7.1	24
	Ω_e^{5h}	10^{-3}	10^5	10^{-5}	45	17.1	33
	Ω_e^{H}	10^{-3}	10^5	10^{-5}	33	24.1	31





yellow: $\alpha = 10^6$ **blue:** $\alpha = \alpha_{\text{min}}$

α_{min}	V_0	tol_{dir}	tol_{tr}	TOL_O	dim V_0	κ	# its.
10^{-2}	V_{GDSW}	–	–	–	33	$2.7 \cdot 10^7$	142
	$V_{\text{DIR\&TR}}$	10^{-}	10^4	10^{-5}	57	7.3	25
1	V_{GDSW}	–	–	–	33	$2.7 \cdot 10^5$	118
	$V_{\text{DIR\&TR}}$	10^{-}	10^4	10^{-5}	57	7.2	25
10^2	V_{GDSW}	–	–	–	33	$2.7 \cdot 10^3$	95
	$V_{\text{DIR\&TR}}$	10^{-}	10^4	10^{-5}	57	7.4	24