Domain decomposition methods for highly heterogeneous problems

Robust coarse spaces and nonlinear preconditioning

Alexander Heinlein[1]

Czech Technical University in Prague, Prague, Czech Republic, February 13, 2023

[1]TU Delft

Based on joint work with Axel Klawonn, Jascha Knepper, Martin Lanser, Janine Weber (University of Cologne), Oliver Rheinbach (TU Bergakademie Freiberg), Kathrin Smetana (Stevens Institute of Technology), and Olof Widlund (New York University)

Outline

1. Schwarz Domain Decomposition Preconditioning

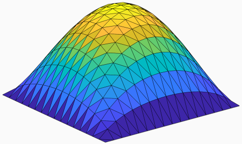
2. Heterogeneous Problems

3. Robust Coarse Spaces for Heterogeneous Problems

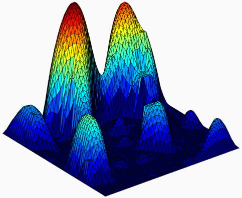
4. Robust Coarse Spaces for Nonlinear Schwarz Preconditioning

Schwarz Domain Decomposition Preconditioning

Solving A Model Problem



$\alpha(x) = 1$



heterogeneous $\alpha(x)$

Consider a **diffusion model problem**:

$$-\nabla \cdot (\alpha(x)\nabla u(x)) = f \quad \text{in } \Omega = [0,1]^2,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Discretization using finite elements yields a **sparse** linear system of equations

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}.$$

Direct solvers

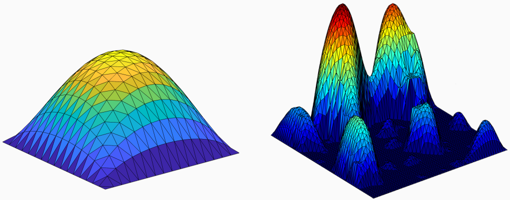
For fine meshes, solving the system using a direct solver is not feasible due to **superlinear complexity and memory cost**.

Iterative solvers

Iterative solvers are efficient for solving sparse linear systems of equations, however, the **convergence rate generally depends on the condition number** $\kappa(\boldsymbol{A})$. It deteriorates, e.g., for

- fine meshes, that is, small element sizes h
- large contrasts $\frac{\max_x \alpha(x)}{\min_x \alpha(x)}$

Solving A Model Problem



$\alpha(x) = 1$ heterogeneous $\alpha(x)$

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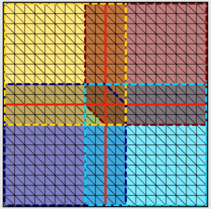
- fine meshes, that is, small element sizes h
- large contrasts $\frac{\max_x \alpha(x)}{\min_x \alpha(x)}$

$\Rightarrow$ We introduce a preconditioner $\boldsymbol{M}^{-1} \approx \boldsymbol{A}^{-1}$ to improve the condition number:

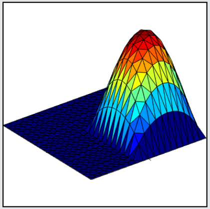
$$\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{u} = \boldsymbol{M}^{-1}\boldsymbol{f}$$

One-level Schwarz preconditioner

Overlap $\delta = 1h$



Solution of local problem



Based on an **overlapping domain decomposition**, we define a **one-level Schwarz operator**

$$M_{\text{OS-1}}^{-1} \boldsymbol{K} = \sum_{i=1}^{N} \boldsymbol{R}_i^T \boldsymbol{K}_i^{-1} \boldsymbol{R}_i \boldsymbol{K},$$

where $\boldsymbol{R}_i$ and $\boldsymbol{R}_i^T$ are restriction and prolongation operators corresponding to Ω_i', and $\boldsymbol{K}_i := \boldsymbol{R}_i \boldsymbol{K} \boldsymbol{R}_i^T$.

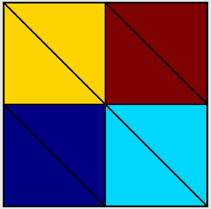
Condition number estimate:

$$\kappa \left(\boldsymbol{M}_{\text{OS-1}}^{-1} \boldsymbol{K} \right) \leq C \left(1 + \frac{1}{H\delta} \right)$$

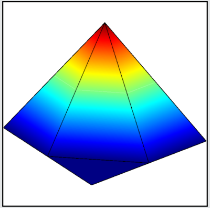
with subdomain size H and overlap width δ.

Lagrangian coarse space

Coarse triangulation



Coarse solution



The **two-level overlapping Schwarz operator** reads

$$\boldsymbol{M}_{\text{OS-2}}^{-1} \boldsymbol{K} = \underbrace{\Phi \boldsymbol{K}_0^{-1} \Phi^T \boldsymbol{K}}_{\text{coarse level -- global}} + \underbrace{\sum_{i=1}^{N} \boldsymbol{R}_i^T \boldsymbol{K}_i^{-1} \boldsymbol{R}_i \boldsymbol{K}}_{\text{first level -- local}},$$

where Φ contains the coarse basis functions and $\boldsymbol{K}_0 := \Phi^T \boldsymbol{K} \Phi$; cf., e.g., Toselli, Widlund (2005).
The construction of a Lagrangian coarse basis requires a coarse triangulation.

Condition number estimate:

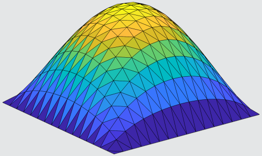
$$\kappa \left(\boldsymbol{M}_{\text{OS-2}}^{-1} \boldsymbol{K} \right) \leq C \left(1 + \frac{H}{\delta} \right)$$

Strengths and Weaknesses of Classical Two-Level Schwarz Preconditioners

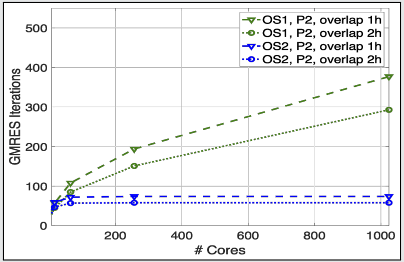
Numerical scalability

Diffusion with **heterogeneous coefficient**:

$$-\Delta u = f \quad \text{in } \Omega = [0,1]^2,$$
$$u = 0 \quad \text{on } \partial\Omega.$$



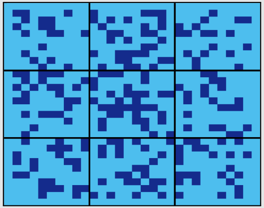
$\#$ subdomains $= \#$ cores, $H/h = 100$



Robustness

Diffusion with **heterogeneous coefficient**:

$$-\nabla \cdot (\alpha(x) \nabla u(x)) = f(x) \quad \text{in } \Omega = [0,1]^2,$$
$$u = 0 \quad \text{on } \partial\Omega.$$



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

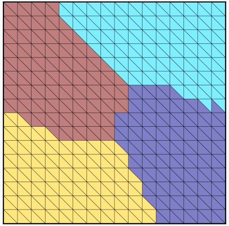
10×10 subdomains with $H/h = 10$ and overlap $1h$

Prec.	its.	κ
–	**>2 000**	$4.51 \cdot 10^8$
$M_{\text{OS-1}}^{-1}$	**>2 000**	$4.51 \cdot 10^8$
$M_{\text{OS-2}}^{-1}$	**586**	$5.56 \cdot 10^5$

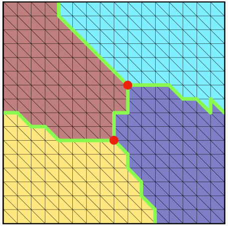
Two-Level Schwarz Preconditioners – GDSW Coarse Space

Instead of a Lagrangian coarse space, we consider a framework based on the **GDSW (Generalized Dryja–Smith–Widlund) coarse space** introduced in Dohrmann, Klawonn, Widlund (2008).

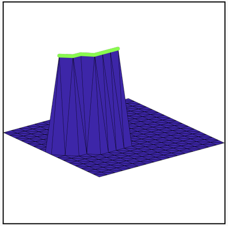
Non-overlapping DD



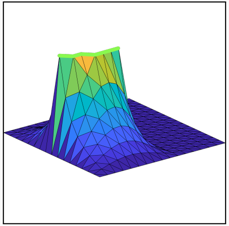
Ident. vertices & edges



Restr. of the null space



Energy minimizing ext.



The coarse basis functions are constructed as **energy minimizing extensions** of functions Φ_Γ that are defined on the interface Γ:

$$\Phi = \begin{bmatrix} -\boldsymbol{A}_{II}^{-1}\boldsymbol{A}_{\Gamma I}^{T}\Phi_\Gamma \\ \Phi_\Gamma \end{bmatrix} = \begin{bmatrix} \Phi_I \\ \Phi_\Gamma \end{bmatrix}$$

The functions Φ_Γ are **restrictions of the null space of global Neumann matrix** to the **edges, vertices, and, in 3D, faces (partition of unity)**.

The **condition number of the GDSW two-level Schwarz operator** is bounded by

$$\kappa\left(\boldsymbol{M}_{\mathrm{GDSW}}^{-1}\boldsymbol{K}\right) \leq C\left(1+\frac{H}{\delta}\right)\left(1+\log\left(\frac{H}{h}\right)\right)^2;$$

cf. Dohrmann, Klawonn, Widlund (2008), Dohrmann, Widlund (2009, 2010, 2012).

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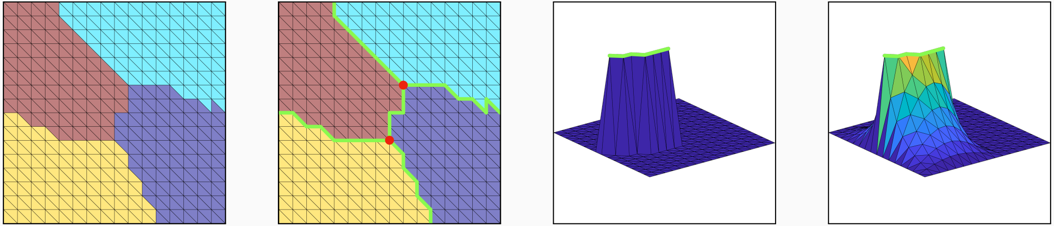
Non-overlapping DD | Ident. vertices & edges | Restr. of the null space | Energy minimizing ext.



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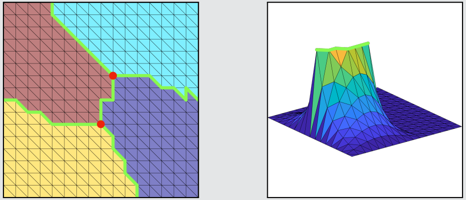
Algebraic approach!

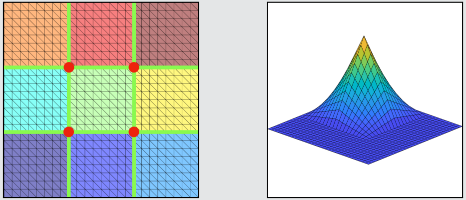
Examples of Extension-Based Coarse Spaces

GDSW (Generalized Dryja–Smith–Widlund)



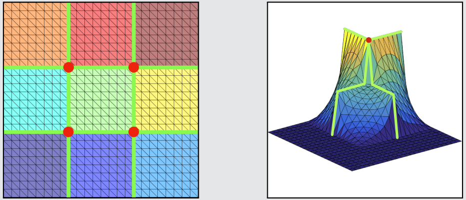
- Dohrmann, Klawonn, Widlund (2008)
- Dohrmann, Widlund (2009, 2010, 2012)

Q1 Lagrangian / piecewise bilinear



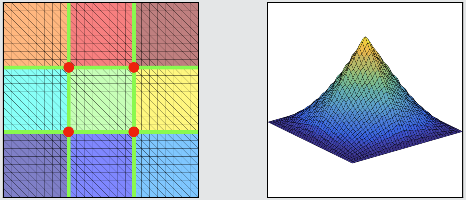
Piecewise linear interface partition of unity functions and a **structured domain decomposition**.

RGDSW (Reduced dimension GDSW)



- Dohrmann, Widlund (2017)
- H., Klawonn, Knepper, Rheinbach, Widlund (2022)

MsFEM (Multiscale Finite Element Method)



- Hou (1997), Efendiev and Hou (2009)
- Buck, Iliev, and Andrä (2013)
- H., Klawonn, Knepper, Rheinbach (2018)

Examples of Extension-Based Coarse Spaces

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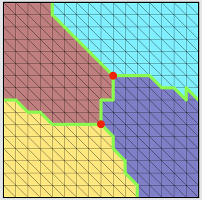

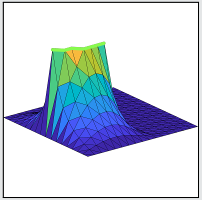

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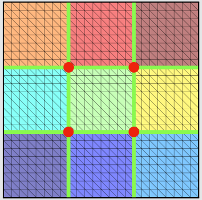

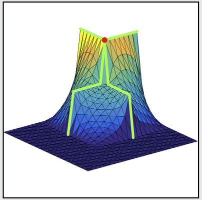

- Dohrmann, Widlund (2017)
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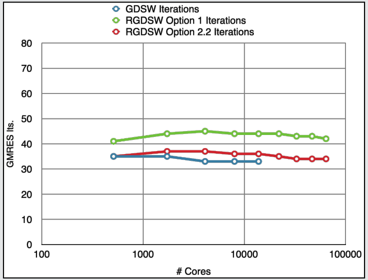
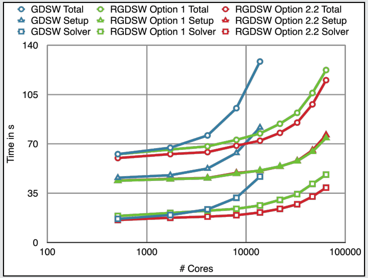
GDSW vs RGDSW

Heinlein, Klawonn, Rheinbach, Widlund (2019).





Examples of Extension-Based Coarse Spaces

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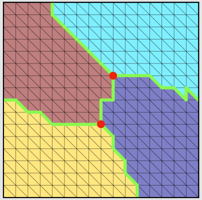
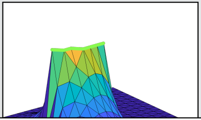
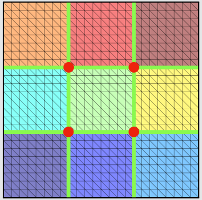
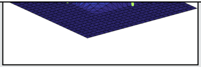




- Dohrmann, Klawonn, Wid
- Dohrmann, Widlund (200

RGDSW (Reduced dimen





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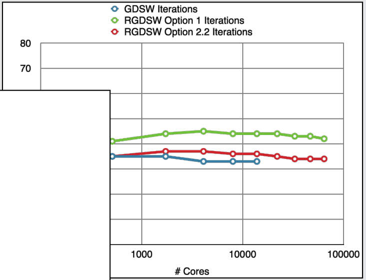
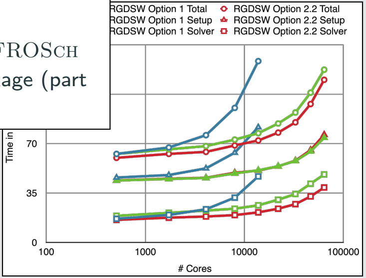
Parallel computations using the FROSCH domain decomposition solver package (part of TRILINOS)

GDSW vs RGDSW

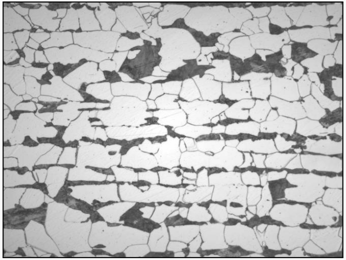
Heinlein, Klawonn, Rheinbach, Widlund (2019).



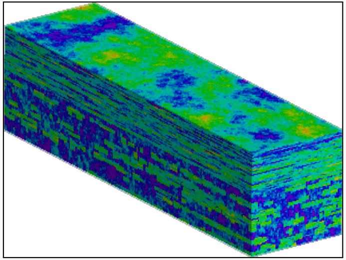


Heterogeneous Problems

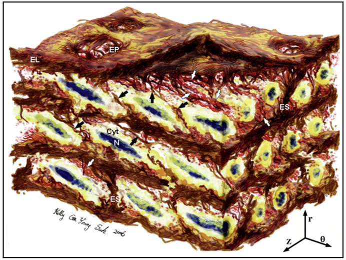
Highly heterogeneous multiscale problems appear in most areas of modern science and engineering, e.g., **composite materials**, **porous media**, and **turbulent transport in high Reynolds number flow**.



Microsection of a dual-phase steel. (Courtesy of Jörg Schröder, University of Duisburg-Essen, Germany; cooperation with ThyssenKrupp Steel.)



Groundwater flow: model 2 from the Tenth SPE Comparative Solution Project; cf. **Christie and Blunt (2001)**.



Representation of the composition of a small segment of arterial walls; taken from **O'Connell et al. (2008)**.

→ The solution of such problems requires a **high spatial and temporal resolution** but also poses **challenges to the solvers**.

Highly Heterogeneous Model Problem

Consider the **diffusion boundary value problem**: find u such that

$$
\begin{aligned}
-\nabla \cdot (\alpha(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega,\\
u &= 0 \qquad\quad \text{on } \partial\Omega,
\end{aligned}
$$

with a **highly varying coefficient function** α. The corresponding weak formulation is: find $u \in H_0^1(\Omega)$, such that

$$a_\Omega(u,v) = f(v) \qquad \forall v \in H_0^1(\Omega)$$

with the bilinear form and linear functional

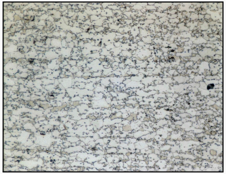
$$a_\Omega(u,v) := \int_\Omega \alpha(x)(\nabla u(x))^T \nabla v(x)\,dx \text{ and } f(v) := \int_\Omega f(x)v(x)\,dx.$$

Discretization using finite elements yields the linear system

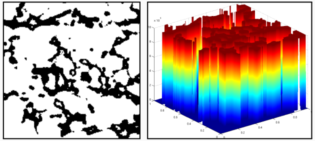
$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$$

with stiffness matrix $\boldsymbol{A}$, discrete solution $\boldsymbol{u}$, and right hand side $\boldsymbol{f}$.

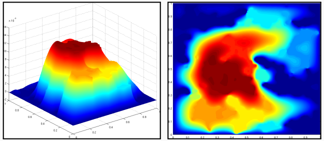
Original microsection of a dual-phase steel



Binary coefficient function



Solution of the BVP

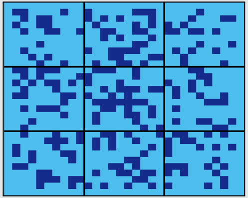


Problem Configuration

Diffusion problem with **random binary coefficient** α: find u such that

$$-\nabla \cdot (\alpha(x)\nabla u(x)) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

Prec.	its.	κ
–	**>2 000**	$4.51 \cdot 10^8$
$M_{\text{OS-1}}^{-1}$	**>2 000**	$4.51 \cdot 10^8$
$M_{\text{OS-2}}^{-1}$	**586**	$5.56 \cdot 10^5$

Observations

→ For **heterogeneous coefficients**, the **condition number clearly deteriorates**. It depends on the **contrast of the coefficient function**

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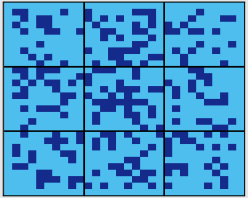

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Observations

→ For **heterogeneous coefficients**, the **condition number clearly deteriorates**. It depends on the **contrast of the coefficient function**

Let us consider some **pathological cases** to better understand the behavior of overlapping Schwarz methods for heterogeneous coefficient distributions.

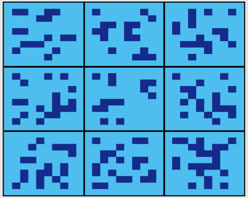
Heterogeneous Problem – Heterogeneities Only Inside Subdomains

Problem Configuration

Diffusion problem with **random binary coefficient α without high coefficients touching the interface**: find u such that

$$-\nabla \cdot (\alpha(x)\nabla u(x)) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



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Prec.	its.	κ
–	$>$**2 000**	$7.99 \cdot 10^8$
$M_{\text{OS-1}}^{-1}$	**64**	133.16
$M_{\text{OS-2}}^{-1}$	**78**	139.15

Observations

$\rightarrow$ In the first level, we **solve the subdomain problems exactly**

$\Rightarrow$ Jumps inside the subdomains are **not problematic**

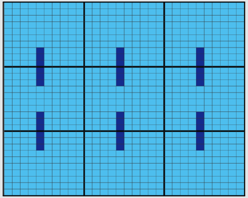
$\rightarrow$ Classical one- and two-level methods are **robust for jumps within the subdomains**

Problem Configuration

Diffusion problem with **binary coefficient α with high contrast channels**: find u such that

$$\begin{aligned} -\nabla \cdot (\alpha(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

Prec.	δ	its.	κ
–		**987**	$8.03 \cdot 10^8$
$M_{\text{OS-1}}^{-1}$	1h	**259**	$83.34 \cdot 10^6$
	2h	**216**	$5.56 \cdot 10^6$
	3h	**37**	91.97
$M_{\text{OS-2}}^{-1}$	1h	**163**	$4.70 \cdot 10^5$
	2h	**128**	$3.24 \cdot 10^5$
	3h	**44**	91.94

Observations

$\rightarrow$ In case the **channels with high coefficient lie completely within the overlapping subdomains**, the method is again **robust**. Otherwise, the convergence **deteriorates**.

$\rightarrow$ In general, it is **not practical to extend the overlap** until each high coefficient component lies **completely within one overlapping subdomain**.

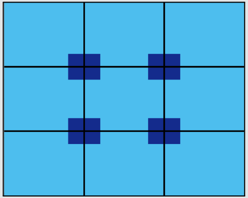
Heterogeneous Problem – Inclusions at the Vertices

Problem Configuration

Diffusion problem with **binary coefficient** α with **high coefficient inclusions at the vertices**: find u such that

$$
\begin{aligned}
-\nabla \cdot (\alpha(x) \nabla u(x)) &= f(x) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

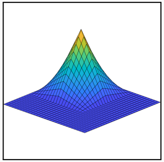
Prec.	its.	κ
–	**874**	$1.35 \cdot 10^9$
$M_{\text{OS-1}}^{-1}$	**163**	$4.06 \cdot 10^7$
$M_{\text{OS-2}}^{-1}$	**138**	$1.07 \cdot 10^6$
M_{MsFEM}^{-1}	**24**	8.05

Observations

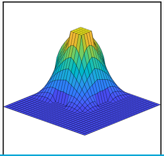
→ In general, one- or two-level Schwarz methods are **not robust** for **high coefficient inclusions at the vertices**

→ **Robustness can be retained** by using **multiscale finite element method (MsFEM)** type functions instead; cf. **Hou (1997), Efendiev and Hou (2009)**

Lagrangian function



MsFEM function

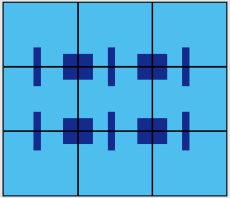


Problem Configuration

Diffusion problem with **binary coefficient** α with **channels and vertex inclusions**: find u such that

$$\begin{aligned} -\nabla \cdot (\alpha(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

Prec.	its.	κ
–	**1708**	$1.16 \cdot 10^9$
$M_{\text{OS-1}}^{-1}$	**447**	$4.17 \cdot 10^7$
$M_{\text{OS-2}}^{-1}$	**268**	$1.10 \cdot 10^6$
M_{MsFEM}^{-1}	**117**	$4.34 \cdot 10^5$

Observations

→ **All of the aforementioned approaches fail** for this example.

→ Since we were able to **deal with the vertex inclusions**, the **problem has to be related to the edges**. How can we construct **suitable coarse basis functions** to deal with **coefficient jumps at the edges**?

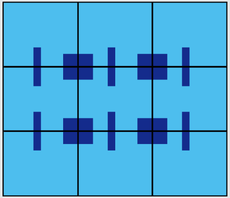
Heterogeneous Problem – Channels & Inclusions

Problem Configuration

Diffusion problem with **binary coefficient** α with **channels and vertex inclusions**: find u such that

$$
\begin{aligned}
-\nabla \cdot (\alpha(x) \nabla u(x)) &= f(x) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Domain decomposition into 10×10 subdomains with $H/h = 10$ and overlap $1h$.



dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

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→ **All of the aforementioned approaches fail** for this example.

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Let us now discuss the **Schwarz theory** in order to construct a **robust coarse space for arbitrary heterogeneous problems**.

Idea of Adaptive Coarse Spaces

Assumption 1: Stable Decomposition

There exists a constant C_0, s.t. for every $\boldsymbol{u} \in V$, there exists a decomposition $\boldsymbol{u} = \sum_{i=0}^{N} \boldsymbol{R}_i^T \boldsymbol{u}_i$, $\boldsymbol{u}_i \in V_i$, with

$$\sum_{i=0}^{N} a_i(\boldsymbol{u}_i, \boldsymbol{u}_i) \leq C_0^2 a(\boldsymbol{u}, \boldsymbol{u}).$$

Assumption 2: Strengthened Cauchy–Schwarz Inequality

There exist constants $0 \leq \epsilon_{ij} \leq 1$, $1 \leq i,j \leq N$, s.t.

$$\begin{aligned}\left|a(\boldsymbol{R}_i^T \boldsymbol{u}_i, \boldsymbol{R}_j^T u_j)\right| \quad & \leq \epsilon_{ij} \quad \left(a(\boldsymbol{R}_i^T \boldsymbol{u}_i, \boldsymbol{R}_i^T \boldsymbol{u}_i)\right)^{1/2} \\ & \qquad\quad \left(a(\boldsymbol{R}_j^T \boldsymbol{u}_j, \boldsymbol{R}_j^T \boldsymbol{u}_j)\right)^{1/2}\end{aligned}$$

for $\boldsymbol{u}_i \in V_i$ and $\boldsymbol{u}_j \in V_j$.
(Consider $\mathcal{E} = (\varepsilon_{ij})$ and $\rho(\mathcal{E})$ its spectral radius)

Assumption 3: Local Stability

There exists $\omega < 0$, such that, for $0 \leq \boldsymbol{u} \neq N$,

$$a(\boldsymbol{R}_i^T \boldsymbol{u}_i, \boldsymbol{R}_i^T \boldsymbol{u}_i) \leq \omega a_i(\boldsymbol{u}_i, \boldsymbol{u}_i), \quad \boldsymbol{u}_i \in \text{range}\left(\tilde{P}_i\right).$$

Idea of spectral coarse spaces

Ensure

$$a(\boldsymbol{u}_0, \boldsymbol{u}_0) \leq C_0^2 a(u, u)$$

by introducing two bilinear forms $c(\cdot,\cdot)$ and $d(\cdot,\cdot)$

$$a(\boldsymbol{u}_0, \boldsymbol{u}_0) \leq C_1 d(\boldsymbol{u}_0, \boldsymbol{u}_0) \quad \textbf{(high energy)}$$

and

$$c(\boldsymbol{u}_0, \boldsymbol{u}_0) \leq C_2 a(\boldsymbol{u}, \boldsymbol{u}), \quad \textbf{(low energy)}$$

where $C_1 C_2$ **is independent of the contrast of the coefficient function** and $u_0 := I_0 u$ is a suitable coarse function.

We **enhance the coarse space by all eigenvectors with eigenvalues below a tolerance** *tol* of

$$d(\boldsymbol{v}, \boldsymbol{w}) = \lambda\, c(\boldsymbol{v}, \boldsymbol{w})$$

and directly obtain

$$\begin{aligned} a(u_0, u_0) \quad & \leq \quad C_1\, d(\boldsymbol{u}_0, \boldsymbol{u}_0) \leq C_1\, tol\, c(\boldsymbol{u}_0, \boldsymbol{u}_0) \\ & \leq \quad C_1\, C_2\, tol\, a(\boldsymbol{u}, \boldsymbol{u}) \end{aligned}$$

In practice, **eigenvalue problem is partitioned into many local eigenvalue problems** $\rightarrow$ parallelization!

Robust Coarse Spaces for Heterogeneous Problems

Adaptive Coarse Spaces in Domain Decomposition Methods – Literature Overview

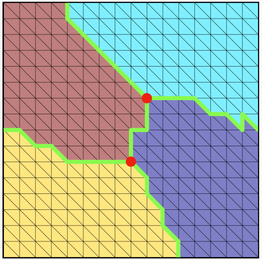
This list is **not exhaustive**:

- **FETI & Neumann–Neumann:** Bjørstad and Krzyzanowski (2002); Bjørstad, Koster, and Krzyzanowski (2001); Rixen and Spillane (2013); Spillane (2015, 2016)
- **BDDC & FETI-DP:** Mandel and Sousedík (2007); Sousedík (2010); Sístek, Mandel, and Sousedík (2012); Dohrmann and Pechstein (2013, 2016); Klawonn, Radtke, and Rheinbach (2014, 2015, 2016); Klawonn, Kühn, and Rheinbach (2015, 2016, 2017); Kim and Chung (2015); Kim, Chung, and Wang (2017); Beirão da Veiga, Pavarino, Scacchi, Widlund, and Zampini (2017); Calvo and Widlund (2016); Oh, Widlund, Zampini, and Dohrmann (2017); Klawonn, Lanser, and Wasiak (preprint 2021)
- **Overlapping Schwarz:** Galvis and Efendiev (2010, 2011); Nataf, Xiang, Dolean, and Spillane (2011); Spillane, Dolean, Hauret, Nataf, Pechstein, and Scheichl (2011); Gander, Loneland, and Rahman (preprint 2015); Eikeland, Marcinkowski, and Rahman (preprint 2016); Heinlein, Klawonn, Knepper, Rheinbach (2018); Marcinkowski and Rahman (2018); Al Daas, Grigori, Jolivet, Tournier (2021); Bastian, Scheichl, Seelinger, and Strehlow (2022); Spillane (preprint 2021, preprint 2021); Bootland, Dolean, Graham, Ma, Scheichl (preprint 2021); Al Daas and Jolivet (preprint 2021)
- Approaches for overlapping Schwarz methods in **this talk**:
  - **AGDSW:** Heinlein, Klawonn, Knepper, Rheinbach (2019, 2019), Heinlein, Klawonn, Knepper, Rheinbach, and Widlund (2022)
  - **Fully Algebraic Coarse Space:** Heinlein and Smetana (Preprint: arXiv:2207.05559)

There is also related work on multigrid methods, such as **AMGe** by Brezina, Cleary, Falgout, Henson, Jones, Manteuffel, McCormick, Ruge (2000).

The **adaptive GDSW (AGDSW) coarse space** is a related approach, which also depends on a **partition of the domain decomposition interface** into edges and vertices. We use

- the **GDSW vertex basis functions** and
- edge functions computed from a **generalized edge eigenvalue problem**.



As a result, the AGDSW coarse space

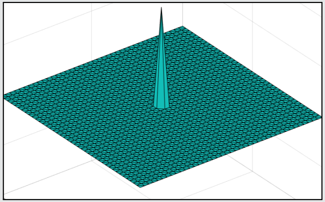
- always **contains the classical GDSW coarse space**.

Cf. **Heinlein, Klawonn, Knepper, Rheinbach (2019, 2019)**.

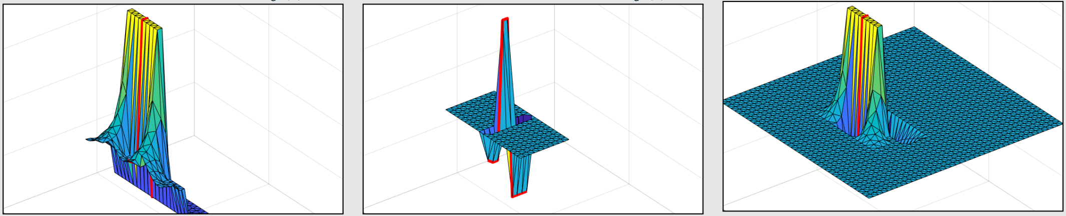
AGDSW vertex basis function

The interior values are then obtained by extending 1 by zero onto the remainder of the interface followed by an energy minimizing extension into the interior:

$$\varphi_v = E_{\Gamma \to \Omega} \left(R_{v \to \Gamma} \left(\mathbb{1}_v \right) \right)$$



AGDSW edge basis functions



Low energy extension $\boldsymbol{E}_{e\to\Omega_e}(\cdot)$ High energy extension $\boldsymbol{R}_{e\to\Omega_e}(\cdot)$ Ext. into the interior

First, we solve the following eigenvalue problem (**in** ***a*****-harmonic space**) for each edge $e \in \mathcal{E}$:

$$a_{\Omega_e}\left(E_{e\to\Omega_e}\left(\tau_{e,*}\right), E_{e\to\Omega_e}\left(\theta\right)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}\left(\tau_{e,*}\right), R_{e\to\Omega_e}\left(\theta\right)\right) \quad \forall \theta \in V_e$$

Then, we select eigenfunctions using the threshold TOL and extend the edge values to Ω:

$$\varphi_{e,*} = E_{\Gamma\to\Omega}\left(R_{e\to\Gamma}\left(\tau_{e,*}\right)\right)$$

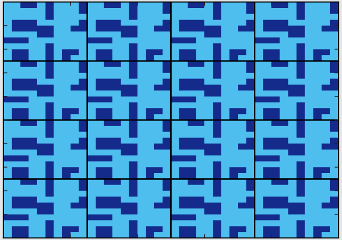
Condition number bound

Using the coarse space $V_{\text{AGDSW}} = \{\varphi_v\} \cup \{\varphi_e\}$ in the two-level Schwarz preconditioner, we obtain

$$\kappa\left(\boldsymbol{M}_{\text{AGDSW}}^{-1}\boldsymbol{K}\right) \leq C\left(1/TOL\right),$$

where C is independent of H, h, and the contrast of the coefficient function α.

Example 1

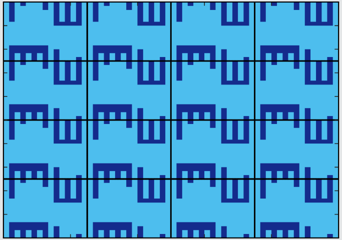


dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

4×4 subdomains, $H/h = 30$, $\delta = 2h$

V_0	*tol*	it.	κ	dim V_0
V_{MsFEM}	-	**199**	$7.8 \cdot 10^5$	**9**
$V_{\text{OS-ACMS}}$	10^{-2}	**23**	5.1	**69**
V_{SHEM}	10^{-3}	**20**	4.3	**69**
V_{AGDSW}	10^{-2}	**29**	7.2	**93**

Example 2



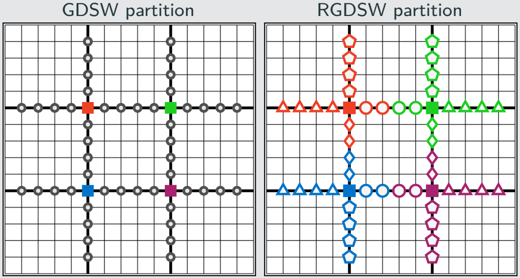
dark blue: $\alpha = 10^8$ **light blue:** $\alpha = 1$

4×4 subdomains, $H/h = 30$, $\delta = 2h$

V_0	*tol*	it.	κ	dim V_0
V_{MsFEM}	-	**282**	$3.8 \cdot 10^7$	9
$V_{\text{OS-ACMS}}$	10^{-2}	**41**	13.2	**33**
V_{SHEM}	10^{-3}	**29**	6.4	**93**
V_{AGDSW}	10^{-2}	**42**	16.5	**45**

SHEM by Gander, Loneland, Rahman (TR 2015), **OS-ACMS** from H., Klawonn, Knepper, Rheinbach (2018), **AGDSW** from H., Klawonn, Knepper, Rheinbach (2019)

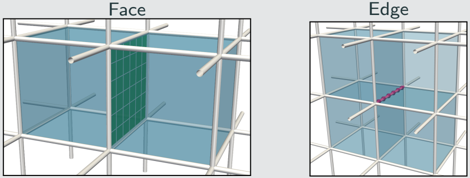
Reducing the coarse space dimension



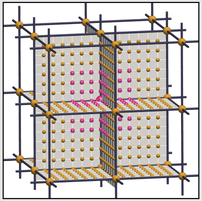
As in the reduced dimension GDSW (RGDSW) approach, we partition the interface into **interface components centered around the vertices**. On these interface components, we solve (slightly modified) eigenvalue problems.

Cf. Heinlein, Klawonn, Knepper, Rheinbach (2021) and Heinlein, Klawonn, Knepper, Rheinbach, Widlund (2022).

Extension to three dimensions

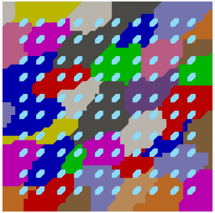


- In AGDSW, we have to solve **face and edge eigenvalue problems**
- In RAGDSW, only the definition of the **interface components changes**

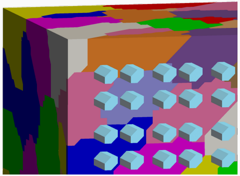


RGDSW interface component

cross section



detailed view of partially peeled beams



V_0	*tol*	iter	κ	dim V_0	$\frac{\dim V_0}{\dim V^h}$
GDSW	–	>2 000	$3.1 \cdot 10^5$	9 996	2.51%
RGDSW	–	>2 000	$3.9 \cdot 10^5$	3 358	0.84%
AGDSW	0.100	71	41.1	14 439	3.63%
AGDSW	0.050	90	59.5	13 945	3.50%
AGDSW	0.010	132	161.1	13 763	3.46%
RAGDSW	0.100	67	34.6	8 249	2.07%
RAGDSW	0.050	88	61.3	7 683	1.93%
RAGDSW	0.010	114	117.4	7 501	1.88%

Heterogeneous linear elasticity problem

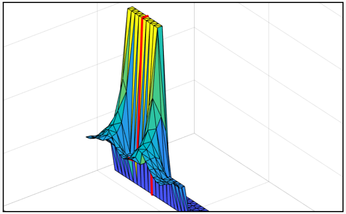
- Ω: cube; Dirichlet boundary condition on $\partial\Omega$.
- Structured tetrahedral mesh; 132 651 nodes (397 953 DOFs); unstructured domain decomposition (METIS); 125 subdomains.
- Poisson ration $\nu = 0.4$.
- Young modulus: elements with $E(T) = 10^6$ in light blue (beams); remainder set to $E(T) = 1$.
- Right hand side $f \equiv 1$.
- Overlap: two layers of finite elements.

- RAGDSW: 45% reduction of coarse space dimension compared to AGDSW (highlighted line).
- RAGDSW: smaller coarse space dimension compared to GDSW and still robust!

Neumann Matrices and Algebraicity

The **low energy property**

$$c(u_0, u_0) \leq C_2 a(u, u)$$



of the bilinear form in the **left hand side of the eigenvalue problems** of AGDSW method is satisfied due to the use of **Neumann boundary conditions**:

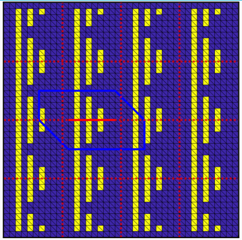
$$a_{\Omega_e}\left(E_{e\to\Omega_e}\left(\tau_{e,*}\right), E_{e\to\Omega_e}\left(\theta\right)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}\left(\tau_{e,*}\right), R_{e\to\Omega_e}\left(\theta\right)\right) \quad \forall \theta \in V_e^0$$

The right hand side matrix just corresponds to the submatrix $\boldsymbol{K}_{ee}$ of $\boldsymbol{K}$ corresponding to the edge e, whereas the Neumann matrices on the left hand sides cannot be extracted from the fully assembled matrix $\boldsymbol{K}$. $\rightarrow$ **not algebraic**

We can make use of the a-orthogonal decomposition

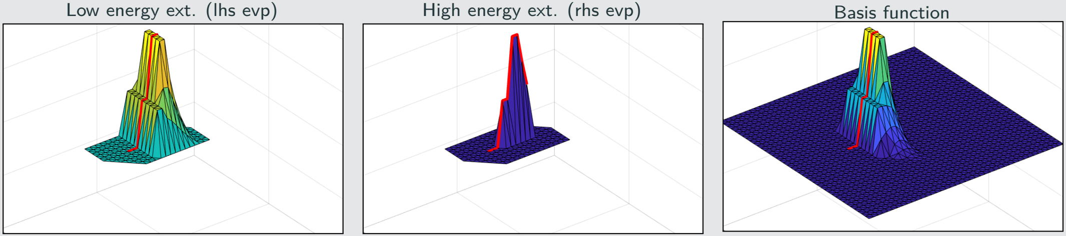
$$V_{\Omega_e} = V^0_{\Omega_e} \oplus \underbrace{\left\{ E_{\partial\Omega_e \to \Omega_e}\left(v\right) : v \in V_{\partial\Omega_e} \right\}}_{=: V_{\Omega_e, \mathrm{harm}}}$$

to ***"split the AGDSW eigenvalue problem"*** into two:



- **Dirichlet eigenvalue problem** on $V^0_{\Omega_e}$
- **Transfer eigenvalue problem** on $V_{\Omega_e,\mathrm{harm}}$; cf. Smetana, Patera (2016)

Dirichlet eigenvalue problem



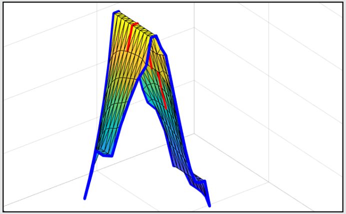
We solve the eigenvalue problem, choose $\lambda_{e,*} < TOL_1$, and extend the basis functions to Ω as before:

$$a_{\Omega_e}\left(E^{\partial\Omega_e}_{e\to\Omega_e}\left(\tau_{e,*}\right), E^{\partial\Omega_e}_{e\to\Omega_e}\left(\theta\right)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e\to\Omega_e}\left(\tau_{e,*}\right), R_{e\to\Omega_e}\left(\theta\right)\right) \quad \forall \theta \in V^0_e$$

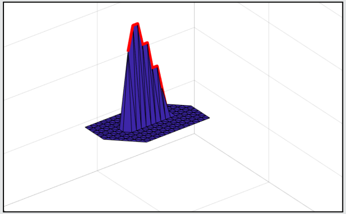
Fully Algebraic Adaptive Coarse Space – Transfer Eigenvalue Problem

Transfer eigenvalue problem

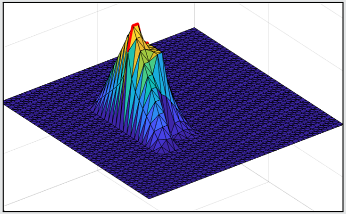
Low energy ext. $E_{\partial\Omega_e \to \Omega_e}(\cdot)$



High energy ext. $R_{e \to \Omega_e}\left(E_{\partial\Omega_e \to \Omega_e}(\cdot)\right)$



Basis function



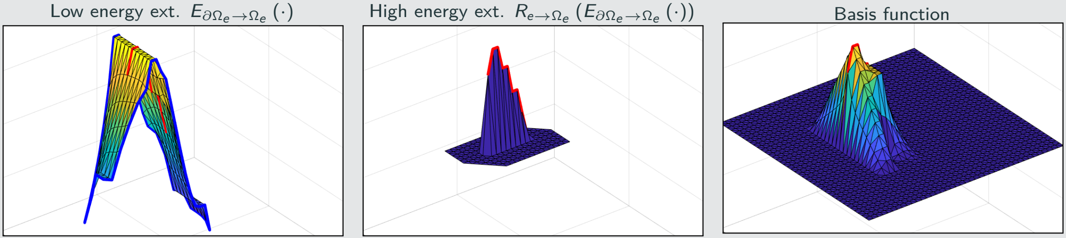
The transfer eigenvalue problem is based on **Smetana, Patera (2016)**. Different from all the eigenvalue problems before, it is solved on the boundary of Ω_e:

$$a_{\Omega_e}\left(E_{\partial\Omega_e \to \Omega_e}(\eta_{e,*}), E_{\partial\Omega_e \to \Omega_e}(\theta)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e \to \Omega_e}\left(E_{\partial\Omega_e \to \Omega_e}(\tau_{e,*})\right), R_{e \to \Omega_e}\left(E_{\partial\Omega_e \to \Omega_e}(\theta)\right)\right) \quad \forall \theta \in V^0_{\partial\Omega_e}$$

We select all eigenfunctions $\eta_{e,*}$ with $\lambda_{e,*}$ above a second **user-chosen threshold** TOL_2. Then, we first compute the edge values $\tau_{e,*} = E_{\partial\Omega_e \to \Omega_e}(\eta_{e,*})\,|_e$ and then extend them into the interior

$$\varphi_{e,*} = E_{\Gamma \to \Omega}\left(R_{e \to \Gamma}(\tau_{e,*})\right)$$

Transfer eigenvalue problem



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$$a_{\Omega_e}\left(E_{\partial\Omega_e \to \Omega_e}(\eta_{e,*}), E_{\partial\Omega_e \to \Omega_e}(\theta)\right) = \lambda_{e,*} a_{\Omega_e}\left(R_{e \to \Omega_e}(E_{\partial\Omega_e \to \Omega_e}(\tau_{e,*})), R_{e \to \Omega_e}(E_{\partial\Omega_e \to \Omega_e}(\theta))\right) \quad \forall \theta \in V^0_{\partial\Omega_e}$$

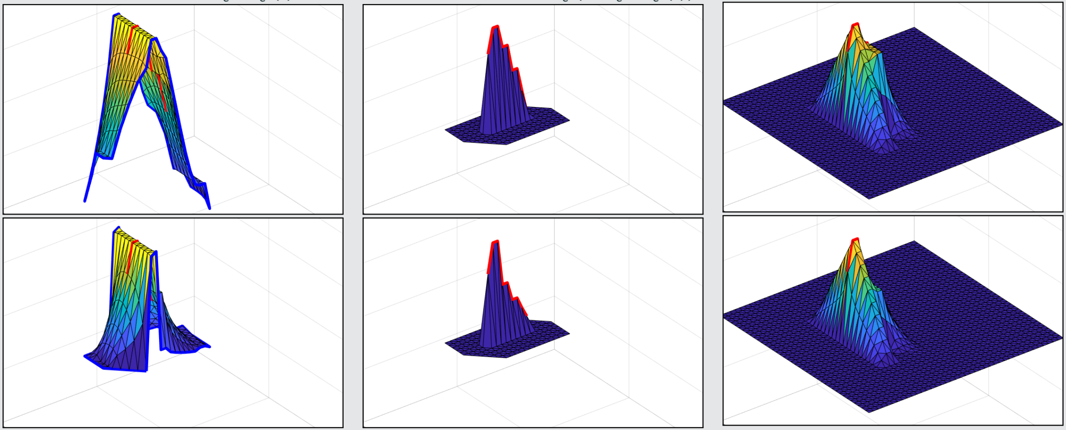
We select all eigenfunctions $\eta_{e,*}$ with $\lambda_{e,*}$ above a second **user-chosen threshold** TOL_2. Then, we first compute the edge values $\tau_{e,*} = E_{\partial\Omega_e \to \Omega_e}(\eta_{e,*})\big|_e$ and then extend them into the interior

$$\varphi_{e,*} = E_{\Gamma \to \Omega}(R_{e \to \Gamma}(\tau_{e,*}))$$

$\rightarrow$ Even though **no Neumann matrices are needed to compute $E_{\partial\Omega_e \to \Omega_e}(\theta)$**, **Neumann matrices are needed to evaluate $a_{\Omega_e}(\cdot,\cdot)$** for functions with nonnegative trace on $\partial\Omega_e$

Algebraic transfer eigenvalue problem

Low energy ext. $E_{\partial\Omega_e \rightarrow \Omega_e}(\cdot)$ High energy ext. $R_{e\rightarrow\Omega_e}\left(E_{\partial\Omega_e \rightarrow \Omega_e}(\cdot)\right)$ Basis function for $a_{\Omega_e}(\cdot,\cdot)$



Low energy ext. $E_{\partial\Omega_e \rightarrow \Omega_e}(\cdot)$ High energy ext. $R_{e\rightarrow\Omega_e}\left(E_{\partial\Omega_e \rightarrow \Omega_e}(\cdot)\right)$ Basis function for $(\cdot,\cdot)_{l_2(\partial\Omega_e)}$

In order to obtain an algebraic transfer eigenvalue problem, we replace $a_{\Omega_e}(\cdot,\cdot)$ by $(\cdot,\cdot)_{l_2(\partial\Omega_e)}$:

$$\left(E_{\partial\Omega_e \rightarrow \Omega_e}(\tau_{e,*}),\, E_{\partial\Omega_e \rightarrow \Omega_e}(\theta)\right)_{l_2(\partial\Omega_e)} = \lambda_{e,*}\, a_{\Omega_e}\left(R_{e\rightarrow\Omega_e}\left(E_{\partial\Omega_e \rightarrow \Omega_e}(\tau_{e,*})\right),\, R_{e\rightarrow\Omega_e}\left(E_{\partial\Omega_e \rightarrow \Omega_e}(\theta)\right)\right) \quad \forall \theta \in V^0_{\partial\Omega_e}$$

Fully Algebraic Adaptive Coarse Space – Condition Number Bound

Condition number estimate (non-algebraic variant)

Using the non-algebraic eigenvalue problem (transfer eigenvalue problem with $a_{\Omega_e}(\cdot,\cdot)$), we obtain a condition number of the form:

$$\kappa\left(\boldsymbol{M}_{\text{DIR\&TR}}^{-1}\boldsymbol{K}\right) \leq C \max\left(\frac{1}{TOL_1}, TOL_2\right),$$

where C is independent of H, h, and the contrast of the coefficient function α.

Condition number estimate (algebraic variant)

Using the algebraic eigenvalue problem (transfer eigenvalue problem with $(\cdot,\cdot)_{l_2(\partial\Omega_e)}$), we obtain a condition number of the form:

$$\kappa\left(\boldsymbol{M}_{\text{DIR\&TR}}^{-1}\boldsymbol{K}\right) \leq C \max\left\{\frac{1}{TOL_1}, \frac{TOL_2}{\alpha_{\min}}\right\},$$

where C is independent of H, h, and the contrast of the coefficient function α.

Cf. **Heinlein and Smetana (Preprint: arXiv:2207.05559)**.

Fully Algebraic Adaptive Coarse Space – Condition Number Bound

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Using the non-algebraic eigenvalue problem (transfer eigenvalue problem with $a_{\Omega_e}(\cdot,\cdot)$), we obtain a condition number of the form:

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Using the algebraic eigenvalue problem (transfer eigenvalue problem with $(\cdot,\cdot)_{l_2(\partial\Omega_e)}$), we obtain a condition number of the form:

$$\kappa\left(\boldsymbol{M}_{\mathrm{DIR\&TR}}^{-1}\boldsymbol{K}\right) \leq C \max\left\{\frac{1}{TOL_1}, \frac{TOL_2}{\alpha_{\mathrm{min}}}\right\},$$

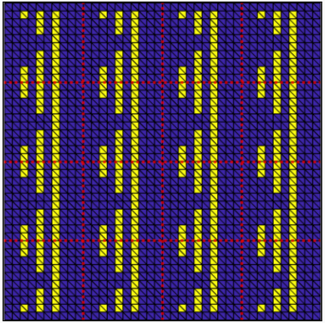
where C is independent of H, h, and the contrast of the coefficient function α.

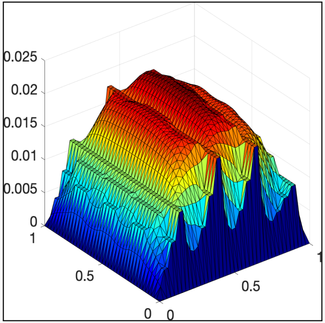
$\rightarrow$ The α_{min} arises from the fact that

$$\frac{h}{N_{\partial\Omega_e}}\alpha_{min}\|\theta\|^2_{l_2(\partial\Omega_e)} \equiv \left|E_{\partial\Omega_e\rightarrow\Omega_e}(\theta)\right|^2_{a,\Omega_e} \quad \forall\theta\in V_{\partial\Omega_e}.$$

Cf. **Heinlein and Smetana (Preprint: arXiv:2207.05559)**.

Numerical Results – Channel Coefficient Function



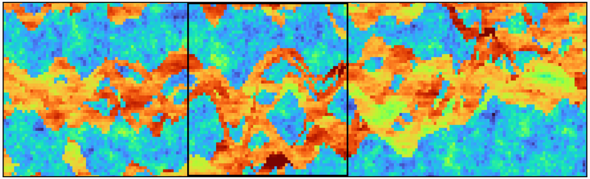


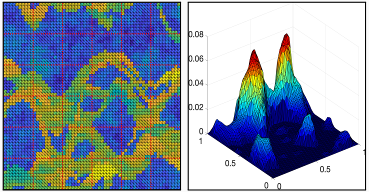
yellow: $\alpha = 10^6$ **blue:** $\alpha = 1$

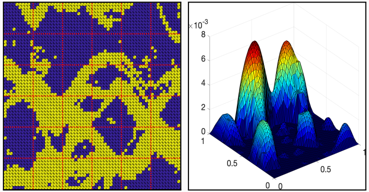
V_0	variant	TOL_{DIR}	TOL_{TR}	TOL_{POD}	dim V_0	κ	# its.
V_{GDSW}	-	-	-	-	33	$2.7 \cdot 10^5$	118
V_{AGDSW}	-	$1.0 \cdot 10^{-2}$			57	7.4	24
$V_{\text{DIR\&TR}}$	$a_{\Omega_e}(\cdot,\cdot)$	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^{1}$	$1.0 \cdot 10^{-5}$	57	7.2	24
$V_{\text{DIR\&TR}}$	$(\cdot,\cdot)_{l_2(\partial\Omega_e)}$	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^{1}$	$1.0 \cdot 10^{-5}$	57	7.2	24

$\rightarrow$ In order to get rid of potential **linear dependencies** between the V_{DIR} and V_{TR} spaces, apply a **proper orthogonal decomposition (POD)** with threshold TOL_{POD} for each edge.

Layer 70 from model 2 of the SPE10 benchmark; cf. Christie and Blunt (2001)





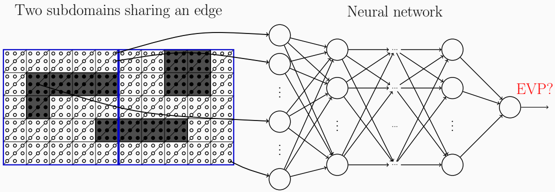


V_0	variant	TOL_{DIR}	TOL_{TR}	TOL_{POD}	dim V_0	κ	# its.
V_{GDSW}	-	-	-	-	85	$2.0 \cdot 10^5$	57
V_{AGDSW}	-	$1.0 \cdot 10^{-2}$			93	19.3	38
$V_{\text{DIR\&TR}}$	$a_{\Omega_e}(\cdot,\cdot)$	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^5$	$1.0 \cdot 10^{-5}$	90	19.4	39
$V_{\text{DIR\&TR}}$	$(\cdot,\cdot)_{l_2(\partial\Omega_e)}$	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^5$	$1.0 \cdot 10^{-5}$	147	9.6	31
Original coefficient $\alpha_{\text{max}} \approx 10^4, \alpha_{\text{min}} \approx 10^{-2}$ (without thresholding)							
V_{GDSW}	-	-	-	-	85	20.6	42

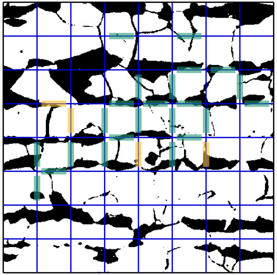
AGDSW & machine learning

Hybrid algorithm: using machine learning techniques in AGDSW.

- **Reduce the computational costs** by detecting all edges (and faces) where local eigenvalue problem have to be solved
- **Samples of the coefficient function** are used as input for a **dense neural network** → **image recognition task**



Binary dual-phase steel microstructure



necessary for robustness
false positive (fp)

→ Approach **originally introduced for adaptive FETI-DP and BDDC**; cf. Heinlein, Lanser, Klawonn, Weber (2019, 2020, 2021, 2021, 2021).

algorithm	τ	cond	it	evp	fp	fn	acc
GDSW	-	3.66e6	$>$ **500**	**0**	-	-	-
AGDSW	-	162.60	**95**	**112**	-	-	-
AGDSW + ML	0.5	9.64e4	**98**	**25**	2	2	95 %
AGDSW + ML	0.45	163.21	**95**	**27**	4	0	95 %

Heinlein, Lanser, Klawonn, Weber (2022)

Observation

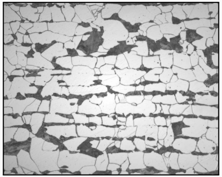
In adaptive FETI-DP or BDDC methods based on Mandel, Sousedík (2005, 2007), for each edge E or face F, a local eigenvalue problem of the form

$$\boldsymbol{v}^T \boldsymbol{P}_D^T \boldsymbol{S} \boldsymbol{P}_D \boldsymbol{w} = \mu \boldsymbol{v}^T \boldsymbol{S} \boldsymbol{w} \quad \forall \boldsymbol{v} \in (\ker \boldsymbol{S})^\perp$$

has to be solved. Here, $\boldsymbol{P}_D$ is a local scaled jump operator and $\boldsymbol{S}$ contains the Schur complement matrices of the subdomains adjacent to E or F. By adding eigenfunctions $\boldsymbol{w}$ with $\mu \geq \mathrm{TOL}$ to the coarse space, we obtain

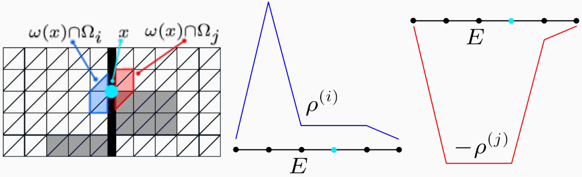
$$\kappa(\boldsymbol{M}^{-1}\boldsymbol{F}) \leq C \cdot \mathrm{TOL};$$

cf. Klawonn, Radtke, Rheinbach (2016), Klawonn, Kühn, Rheinbach (2016).



Microsection of a dual-phase steel.
Courtesy of J. Schröder.

$$\begin{aligned} -\nabla \cdot (\rho(x)\nabla u(x)) &= f(x) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$



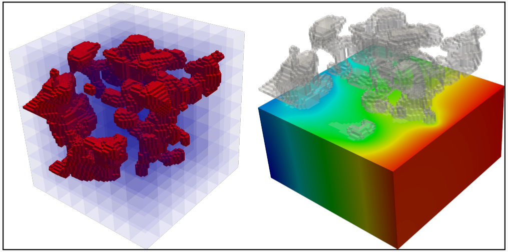
Approach

By constructing coarse basis functions w_{fr} with large values for

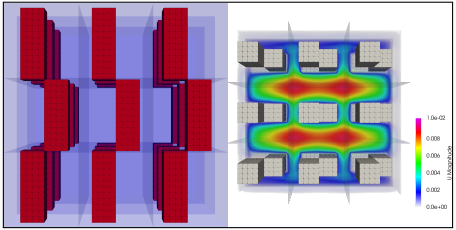
$$\frac{w_{\mathrm{fr}}^T P_D^T S P_D w_{\mathrm{fr}}}{w_{\mathrm{fr}}^T S w_{\mathrm{fr}}}$$

using the coefficient function ρ, we obtain functions which are **close** the adaptive coarse space. $\Rightarrow$ **Robust and efficient coarse space.**

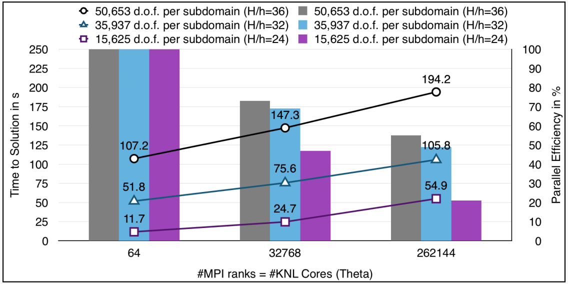
approach	# c.	cond	it	TtS
coefficient jump $1e+3$; $H/h = 24$				
frugal	9 093	1.67e+2	**76**	**123.8s**
face-avg	5 061	1.19e+3	274	275.6s
face-avg & rot	9 093	5.09e+2	179	211.7s
coefficient jump $1e+6$; $H/h = 24$				
approach	**# c.**	**cond**	**it**	**TtS**
frugal	9 093	2.44e+4	**179**	**210.9s**
face-avg	5 061	9.73e+5	>1000	>893.7s
face-avg & rot	9 093	4.70e+5	>1000	>924.9s



Dual-phase steel RVE with linear elasticity; 8^3 subdomains.



Heterogeneous diffusion with coefficient 10^6



Parallel simulations on magnitUDE (UDUE) / Theta (ANL); cf. **Heinlein, Klawonn, Lanser, Weber (2020)**.

Robust Coarse Spaces for Nonlinear Schwarz Preconditioning

Let us consider the nonlinear problem arising from the discretization of a partial differential equation

$$\boldsymbol{F}(\boldsymbol{u}) = 0.$$

We solve the problem using a **Newton-Krylov approach**, i.e., we solve a sequence of linearized problems using a Krylov subspace method:

$$\boldsymbol{DF}\left(\boldsymbol{u}^{(k)}\right) \Delta \boldsymbol{u}^{(k+1)} = \boldsymbol{F}\left(\boldsymbol{u}^{(k)}\right).$$

Linear preconditioning

In linear preconditioning, we **improve the convergence speed of the linear solver** by constructing a **linear operator** M^{-1} and solve linear systems

$$\boldsymbol{M}^{-1}\boldsymbol{DF}\left(\boldsymbol{u}^{(k)}\right) \Delta \boldsymbol{u}^{(k+1)} = \boldsymbol{M}^{-1}\boldsymbol{F}(\boldsymbol{u}^{(k)}).$$

Goal:

- $\kappa\left(\boldsymbol{M}^{-1}\boldsymbol{DF}\left(\boldsymbol{u}^{(k)}\right)\right) \approx 1.$

$\Rightarrow \boldsymbol{M}^{-1}\boldsymbol{DF}\left(\boldsymbol{u}^{(k)}\right) \approx \boldsymbol{I}.$

Nonlinear preconditioning

In nonlinear preconditioning, we **improve the convergence speed of the nonlinear solver** by constructing a **nonlinear operator** G and solve the nonlinear system

$$(\boldsymbol{G} \circ \boldsymbol{F})(\boldsymbol{u}) = 0.$$

Goals:

- $\boldsymbol{G} \circ \boldsymbol{F}$ almost linear.
- Additionally: $\kappa\left(\boldsymbol{D}(\boldsymbol{G} \circ \boldsymbol{F})(\boldsymbol{u})\right) \approx 1.$

Additive nonlinear left preconditioners (based on Schwarz methods)

ASPIN/ASPEN: Cai, Keyes 2002; Cai, Keyes, Marcinkowski (2002); Hwang, Cai (2005, 2007); Groß, Krause (2010, 2013)

RASPEN: Dolean, Gander, Kherijii, Kwok, Masson (2016)

MSPIN: Keyes, Liu, (2015, 2016, 2021); Liu, Wei, Keyes (2017)

Two-Level nonlinear Schwarz: Heinlein, Lanser (2020); Heinlein, Lanser, Klawonn (2022)

Nonlinear right preconditioners

Nonlinear FETI-DP/BDDC: Klawonn, Lanser, Rheinbach (2012, 2013, 2014, 2015, 2016, 2018); Klawonn, Lanser, Rheinbach, Uran (2017, 2018)

Nonlinear Elimination: Hwang, Lin, Cai (2010); Cai, Li (2011); Wang, Su, Cai (2015); Hwang, Su, Cai (2016); Gong, Cai (2018); Luo, Shiu, Chen, Cai (2019); Gong, Cai (2019)

Nonlinear Neumann-Neumann: Bordeu, Boucard, Gosselet (2009)

Nonlinear FETI-1: Pebrel, Rey, Gosselet (2008); Negrello, Gosselet, Rey (2021)

Other DD work reversing linearization and decomposition: Ganis, Juntunen, Pencheva, Wheeler, Yotov (2014); Ganis, Kumar, Pencheva, Wheeler, Yotov (2014)

Early nonlinear DD work: Cai, Dryja (1994); Dryja, Hackbusch (1997)

ASPEN & ASPIN

Our approach is based on the nonlinear one-level Schwarz methods **ASPEN (Additive Schwarz Preconditioned Exact Newton)** and **ASPIN (Additive Schwarz Preconditioned Inexact Newton)** introduced in Cai and Keyes (2002). The nonlinear finite element problem

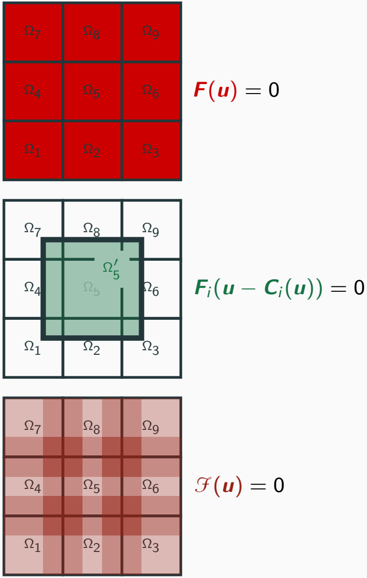
$$\boldsymbol{F}(\boldsymbol{u}) = 0 \quad \text{with } \boldsymbol{F} : V \to V$$

is reformulated to

$$\mathcal{F}(\boldsymbol{u}) = \boldsymbol{G}(\boldsymbol{F}(\boldsymbol{u})) = 0.$$

The **nonlinear left-preconditioner $\boldsymbol{G}$ is only given implicitly** by **solving the nonlinear problem locally** on each of the (overlapping) subdomains. Roughly,

$$\boldsymbol{F}_i(\boldsymbol{u} - \underbrace{\boldsymbol{C}_i(\boldsymbol{u})}_{\text{local correction}}), \; i = 1, \ldots, N.$$



ASPEN

Local corrections $\boldsymbol{T}_i(\boldsymbol{u})$:

$$\boldsymbol{R}_i \boldsymbol{F}(\boldsymbol{u} - \boldsymbol{P}_i \boldsymbol{T}_i(\boldsymbol{u})) = 0, \;\; i = 1, \ldots, N, \;\; \text{with}$$

$$\text{restrictions} \quad \boldsymbol{R}_i : V \to V_i,$$
$$\text{prolongations} \quad \boldsymbol{P}_i : V_i \to V.$$

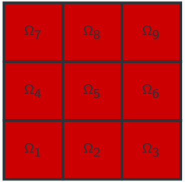
Nonlinear ASPEN problem:

$$\mathcal{F}_A(\boldsymbol{u}) := \sum\nolimits_{i=1}^{N} \boldsymbol{P}_i \boldsymbol{T}_i(\boldsymbol{u}) = 0$$

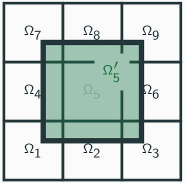
We solve $\mathcal{F}_A(\boldsymbol{u}) = 0$ using Newton's method with $u_i = u - \boldsymbol{P}_i \boldsymbol{T}_i(\boldsymbol{u})$. The Jacobian writes

$$D\mathcal{F}_A(\boldsymbol{u}) = \sum\nolimits_{i=1}^{N} \underbrace{\boldsymbol{P}_i \left(\boldsymbol{R}_i D\boldsymbol{F}(\boldsymbol{u}_i) P_i\right)^{-1} \boldsymbol{R}_i \; D\boldsymbol{F}(\boldsymbol{u}_i)}_{\substack{\text{local Schwarz operators} \\ \text{(preconditioned operators)}}}$$

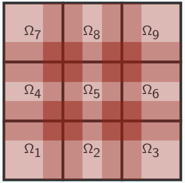
- $\boldsymbol{F}(\boldsymbol{u}) = 0 \Leftrightarrow \mathcal{F}_A(\boldsymbol{u}) = 0$ near a given solution
- $D\boldsymbol{F}(\boldsymbol{u}_i)$ **global** but can be **assembled locally**



$\boldsymbol{F}(\boldsymbol{u}) = 0$



$\boldsymbol{R}_i \boldsymbol{F}(\boldsymbol{u} - \boldsymbol{R}_i^T \boldsymbol{T}_i(\boldsymbol{u})) = 0$



$\sum_{i=1}^{N} \boldsymbol{R}_i^T \boldsymbol{T}_i(\boldsymbol{u}) = 0$

Nonlinear One-Level Schwarz Preconditioners

RASPEN (Dolean et al. (2016))

Local corrections $T_i(u)$:

$$R_i F(u - P_i T_i(u)) = 0, \; i = 1, ..., N, \text{ with}$$

restrictions $R_i : V \to V_i$,

prolongations $P_i, \widetilde{P}_i : V_i \to V$.

Nonlinear RASPEN problem:

$$\mathcal{F}_{RA}(u) := \sum_{i=1}^{N} \widetilde{P}_i T_i(u) = 0$$

We solve $\mathcal{F}_{RA}(u) = 0$ using Newton's method with $u_i = u - P_i T_i(u)$. The Jacobian writes

$$D\mathcal{F}_{RA}(u) = \sum_{i=1}^{N} \underbrace{\widetilde{P}_i \left(R_i DF(u_i) P_i\right)^{-1} R_i \; DF(u_i)}_{\substack{\text{local Schwarz operators} \\ \text{(preconditioned operators)}}}$$

- $\sum_{i=1}^{N} \widetilde{P}_i R_i = I$
- **Reduced communication** & (often) **better conv.**

Results

***p*-Laplacian model problem**

$$\begin{aligned} -\alpha \Delta_p u &= 1 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

with $\alpha \Delta_p u := \mathrm{div}(\alpha |\nabla u|^{p-2} \nabla u)$.

$p = 4$; $H/h = 16$; overlap $\delta = 1$				
N	**solver**	**nonlin.** outer it.	**nonlin.** inner it. (avg.)	**lin.** GMRES it. (sum)
9	NK-RAS	18	-	272
	RASPEN	5	25.2	89
25	NK-RAS	19	-	488
	RASPEN	6	28.3	172
49	NK-RAS	20	-	691
	RASPEN	6	27.3	232

$\Rightarrow$ **Improved nonlinear convergence**, but **no scalability** in the linear iterations.

Two-level (R)ASPEN (Heinlein & Lanser (2020))

Local/Coarse corrections $T_i(\boldsymbol{u})$:

$$R_i F(u - P_i T_i(\boldsymbol{u})) = 0,\ i = 0, 1, \ldots, N,\ \text{with}$$

$$\text{restrictions}\quad \boldsymbol{R}_i : V \to V_i,$$
$$\text{prolongations}\quad \boldsymbol{P}_i : V_i \to V.$$

Nonlinear two-level ASPEN problem:

$$\mathcal{F}_A(\boldsymbol{u}) := \boldsymbol{P}_0 \boldsymbol{T}_0(\boldsymbol{u}) + \sum\nolimits_{i=1}^{N} \boldsymbol{P}_i \boldsymbol{T}_i(\boldsymbol{u}) = 0$$

We solve $\mathcal{F}_A(\boldsymbol{u}) = 0$ using Newton's method with $\boldsymbol{u}_i = \boldsymbol{u} - \boldsymbol{P}_i \boldsymbol{T}_i(\boldsymbol{u})$. The Jacobian writes

$$\begin{aligned} D\mathcal{F}_{RA}(\boldsymbol{u}) \quad = \quad & \overbrace{P_0 \left(\boldsymbol{R}_0 \boldsymbol{DF}(\boldsymbol{u}_0) \boldsymbol{P}_0\right)^{-1} R_0\ DF(u_0)}^{\text{coarse Schwarz operator}} \\ & + \sum\nolimits_{i=1}^{N} \underbrace{P_i \left(\boldsymbol{R}_i \boldsymbol{DF}(\boldsymbol{u}_i) \boldsymbol{P}_i\right)^{-1} \boldsymbol{R}_i\ \boldsymbol{DF}(\boldsymbol{u}_i)}_{\text{local Schwarz operators}} \end{aligned}$$

Results for p-Laplace

1-lvl One-level RASPEN

2-lvl A Two-level RASPEN with **additively coupled coarse level**

2-lvl M Two-level RASPEN with **multiplicatively coupled coarse level**

$p = 4$; $H/h = 16$; overlap $\delta = 1$						
		nonlin.			**lin.**
N	RASPEN **solver**	**outer it.**	**inner it. (avg.)**	**coarse it.**	GMRES **it. (sum)**
	1-lvl	5	25.2	-	89
9	2-lvl A	6	33.4	27	93
	2-lvl M	4	17.1	29	52
	1-lvl	6	27.3	-	232
49	2-lvl A	6	29.2	28	137
	2-lvl M	4	12.6	29	80

$\Rightarrow$ **Improved nonlinear convergence and scalability**.

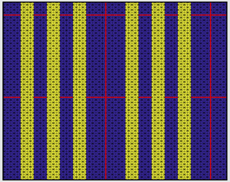
Problem configuration (Heinlein, Klawonn, Lanser (2022))

p-Laplacian problem with $p = 4$ and a **binary coefficient** α:

find u such that

$$\begin{aligned} -\alpha\Delta_p u &= 1 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Domain decomposition into 6×6 subdomains with $H/h = 32$ and overlap $1h$.



yellow: $\alpha = 10^3$ **blue:** $\alpha = 1$

size cp	method	coarse space	outer it.	local it. (avg.)	coarse it.	GMRES it. (sum)
no globalization						
145	H1-RASPEN	AGDSW	5	27.0	35	77
25	H1-RASPEN	MsFEM-D	>20	-	-	-
25	H1-RASPEN	MsFEM-E	>20	-	-	-
145	NK-RAS	AGDSW	>20	-	-	-
inexact Newton backtracking (INB); cf. Eisenstat and Walker (1994)						
145	H1-RASPEN	AGDSW	5	24.8	21	77
25	H1-RASPEN	MsFEM-D	**15**	**75.8**	**62**	**645**
25	H1-RASPEN	MsFEM-E	**18**	**83.9**	**75**	**852**
145	NK-RAS	AGDSW	**13**	-	-	207

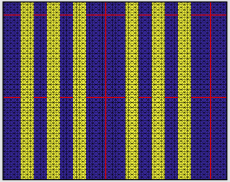
Problem configuration (Heinlein, Klawonn, Lanser (2022))

p-Laplacian problem with $p = 4$ and a **binary coefficient** α:

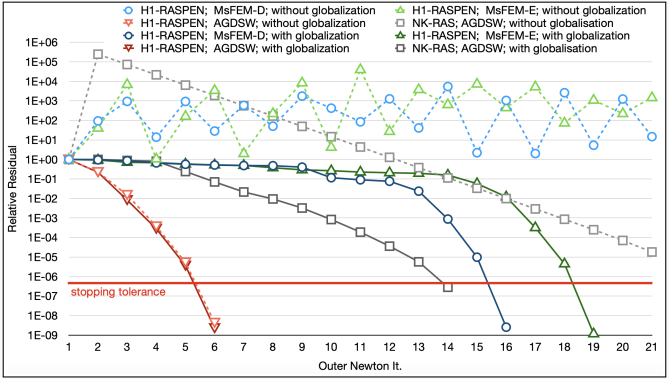
find u such that

$$\begin{aligned} -\alpha\Delta_p u &= 1 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Domain decomposition into 6×6 subdomains with $H/h = 32$ and overlap $1h$.



yellow: $\alpha = 10^3$ **blue:** $\alpha = 1$

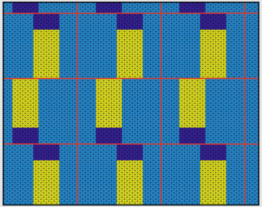


Problem configuration (Heinlein, Klawonn, Lanser (2022))

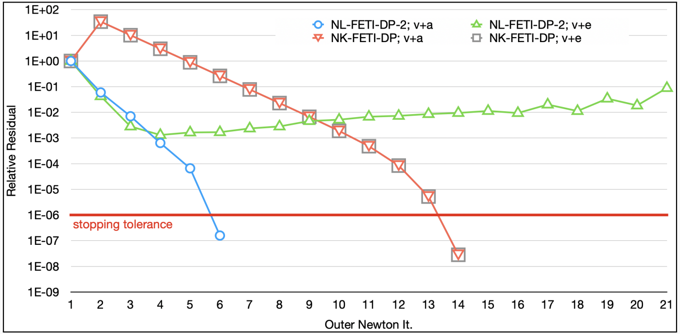
p-Laplacian problem and a **binary coefficient** α: find u such that

$$\begin{aligned} -\alpha \Delta_p u &= 1 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Domain decomposition into 6×6 subdomains with $H/h = 32$ and overlap $1h$.



yellow: $\alpha = 10^3$, $p = 4$ **blue:** $\alpha = 1$, $p = 4$
light blue: $\alpha = 1$, $p = 2$



Thank you for your attention!

Summary

- Robustness of domain decomposition preconditioners for **highly heterogeneous problems** generally require special treatment. One effective approach is the use of **robust coarse spaces**, for instance, using **local generalized eigenvalue problems**.
- **Newton convergence for nonlinear problems** (as well as the linear convergence in each linearization step) can be **significantly improved using nonlinear domain decomposition methods**.
- For **highly heterogeneous nonlinear problems**, (only) the **combination of nonlinear preconditioning and robust coarse spaces** may ensure a **robust solver framework**.

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