



SCaLA



Advances of FROSch Preconditioners for Multiphysics and Multiscale Simulations

Alexander Heinlein¹

29th International Conference on Domain Decomposition Methods, July 23-27, 2025

¹Delft University of Technology

1 The FROSch Package – Algebraic and Parallel Schwarz Preconditioners in TRILINOS

2 Monolithic Coarse Spaces Multiphysics Problems

Based on joint work with

**Axel Klawonn, Jascha Knepper, and
Lea Saßmannshausen**

(University of Cologne)

**Mauro Perego, Siva Rajamanickam, and
Ichitaro Yamazaki**

(Sandia National Laboratories)

3 Robust Coarse Spaces for Heterogeneous Problems

Based on joint work with

Filipe Cumarú and Hadi Hajibeygi

(Delft University of Technology)

Axel Klawonn and Jascha Knepper

(University of Cologne)

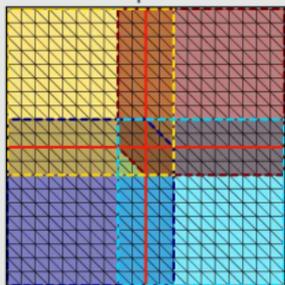
Ichitaro Yamazaki

(Sandia National Laboratories)

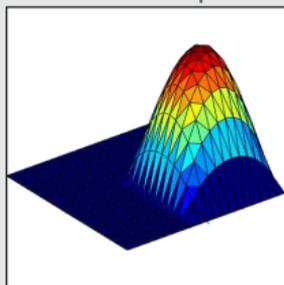
The FROSch Package – Algebraic and Parallel Schwarz Preconditioners in Trilinos

One-level Schwarz preconditioner

Overlap $\delta = 2h$



Solution of local problem



Based on an **overlapping domain decomposition**, we define a **one-level Schwarz operator**

$$M_{OS-1}^{-1} \mathbf{A} = \sum_{i=1}^N \mathbf{R}_i^{\top} \mathbf{A}_i^{-1} \mathbf{R}_i \mathbf{A},$$

where \mathbf{R}_i and \mathbf{R}_i^{\top} are restriction and prolongation operators corresponding to Ω'_i , and $\mathbf{A}_i := \mathbf{R}_i \mathbf{A} \mathbf{R}_i^{\top}$.

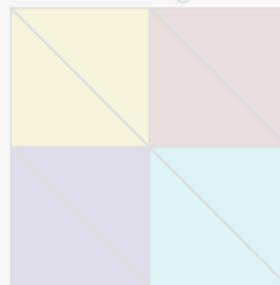
Condition number estimate:

$$\kappa \left(M_{OS-1}^{-1} \mathbf{A} \right) \leq C \left(1 + \frac{1}{H\delta} \right)$$

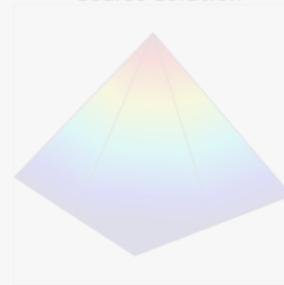
with subdomain size H and overlap width δ .

Lagrangian coarse space

Coarse triangulation



Coarse solution



The two-level overlapping Schwarz operator reads

$$M_{OS-2}^{-1} \mathbf{A} = \underbrace{\Phi \mathbf{A}_0^{-1} \Phi^{\top} \mathbf{A}}_{\text{coarse level - global}} + \sum_{i=1}^N \underbrace{\mathbf{R}_i^{\top} \mathbf{A}_i^{-1} \mathbf{R}_i \mathbf{A}}_{\text{first level - local}},$$

where Φ contains the coarse basis functions and $\mathbf{A}_0 := \Phi^{\top} \mathbf{A} \Phi$; cf., e.g., **Toselli, Widlund (2005)**.

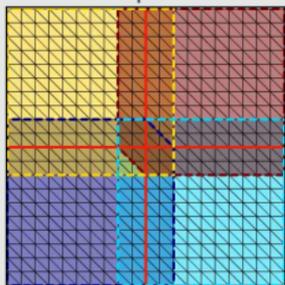
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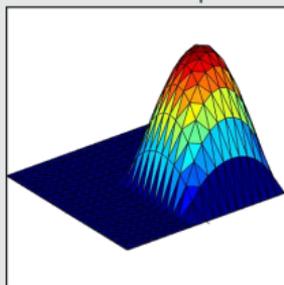
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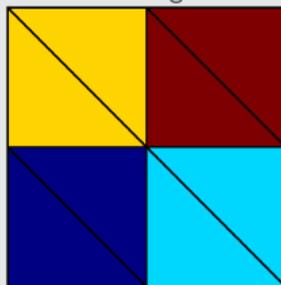
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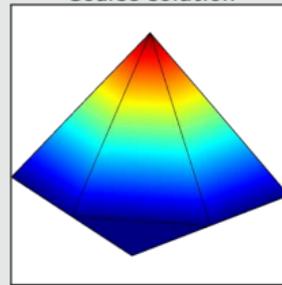
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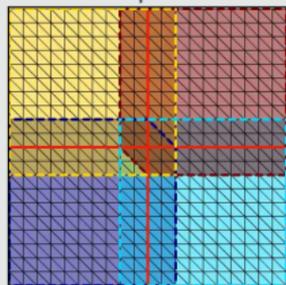
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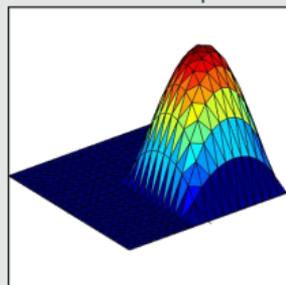
Two-Level Schwarz Preconditioners

One-level Schwarz preconditioner

Overlap $\delta = 2h$

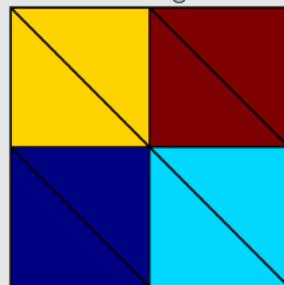


Solution of local problem

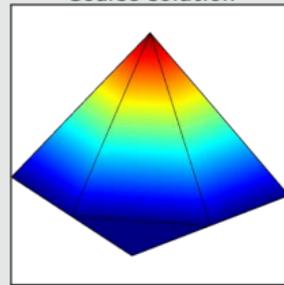


Lagrangian coarse space

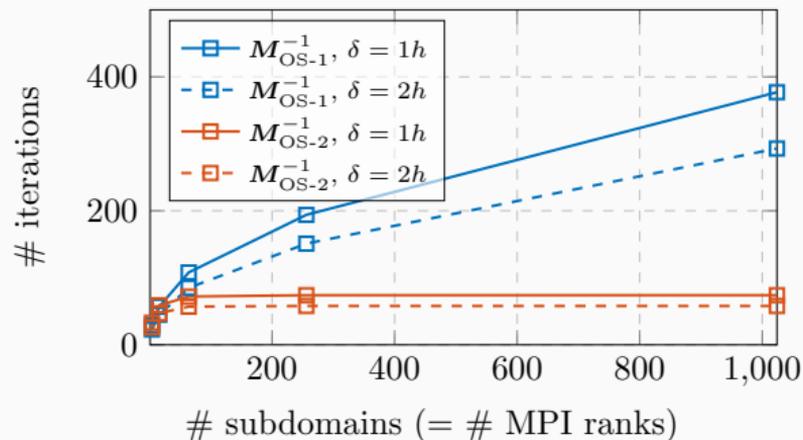
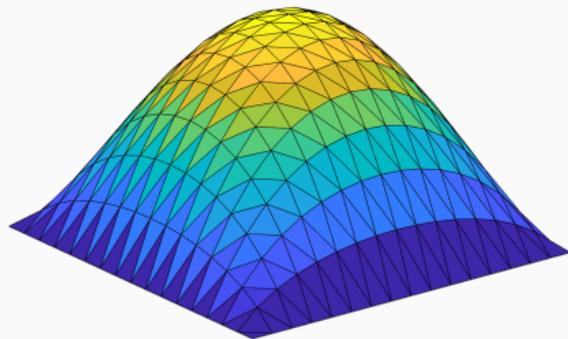
Coarse triangulation



Coarse solution



Diffusion model problem in two dimensions,
 $H/h = 100$



FROSch (Fast and Robust Overlapping Schwarz) Framework in Trilinos



Sandia
National
Laboratories



TUBAF
Die Ressourcenuniversität.
Seit 1765.

Software

- Object-oriented C++ domain decomposition solver framework with MPI-based distributed memory parallelization
- Part of TRILINOS with the parallel linear algebra based on TPETRA
- Node-level parallelization and performance portability on CPU and GPU architectures through KOKKOS and KOKKOSKERNELS
- Accessible through unified TRILINOS solver interface STRATIMIKOS

Methodology

- **Parallel scalable multi-level Schwarz domain decomposition preconditioners**
- **Algebraic construction** based on the parallel distributed system matrix
- **Extension-based coarse spaces**

Team (active)

- Filipe Cumaru (TU Delft)
- Alexander Heinlein (TU Delft)
- Kyrill Ho (UCologne)
- Sebastian Kinnewig (LUH)
- Axel Klawonn (UCologne)
- Jascha Knepper (UCologne)
- Stephan Köhler (TUBAF)
- Friederike Röver (TUBAF)
- Siva Rajamanickam (SNL)
- Oliver Rheinbach (TUBAF)
- Lea Saßmannshausen (UCologne)
- Ichitaro Yamazaki (SNL)

Partition of Unity

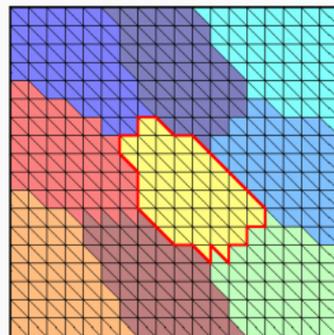
The **energy-minimizing extension** $v_i = H_{\partial\Omega_i \rightarrow \Omega_i}(v_{i,\partial\Omega_i})$ solves

$$\begin{aligned} -\Delta v_i &= 0 && \text{in } \Omega_i, \\ v_i &= v_{i,\partial\Omega_i} && \text{on } \partial\Omega_i. \end{aligned}$$

Hence, $v_i = E_{\partial\Omega_i \rightarrow \Omega_i}(\mathbb{1}_{\partial\Omega_i}) = \mathbb{1}$.

Due to **linearity of the extension operator**, we have

$$\sum_i \varphi_i = \mathbb{1}_{\partial\Omega_i} \Rightarrow \sum_i E_{\partial\Omega_i \rightarrow \Omega_i}(\varphi_i) = \mathbb{1}_{\Omega_i}$$



Null space property

Any extension-based coarse space built from a partition of unity on the domain decomposition interface satisfies the **null space property necessary for numerical scalability**:



Algebraicity of the energy-minimizing extension

The computation of energy-minimizing extensions only requires K_{II} and $K_{I\Gamma}$, **submatrices of the fully assembled matrix K_i** .

$$\mathbf{v} = \begin{bmatrix} -K_{II}^{-1} K_{I\Gamma} \\ I_{\Gamma} \end{bmatrix} \mathbf{v}_{\Gamma},$$

Overlapping domain decomposition

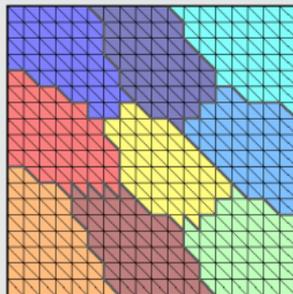
The **overlapping subdomains** are constructed by **recursively adding layers of elements** via the sparsity pattern of K .

The corresponding matrices

$$K_i = R_i K R_i^T$$

can easily be extracted from K .

Nonoverlapping DD



Overlapping domain decomposition

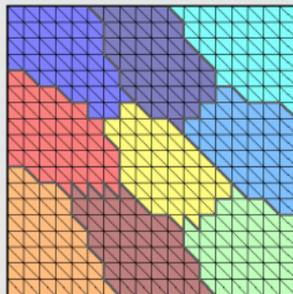
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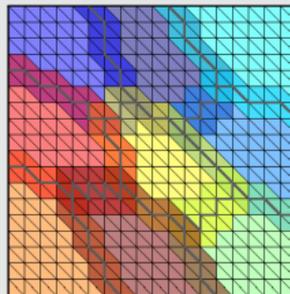
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Overlap $\delta = 1h$



Overlapping domain decomposition

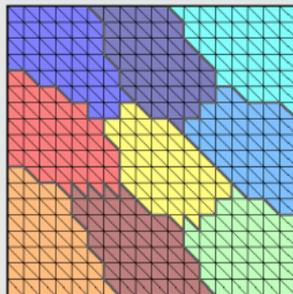
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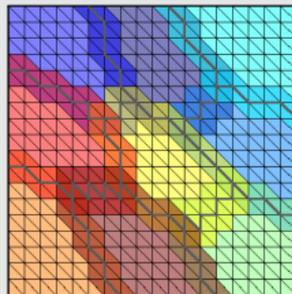
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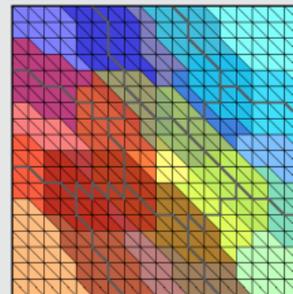
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Overlap $\delta = 2h$



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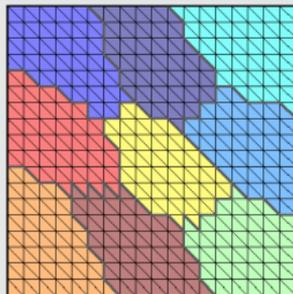
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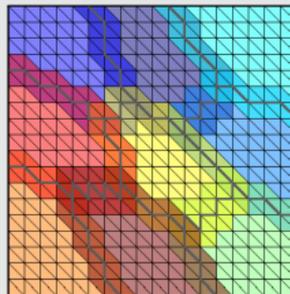
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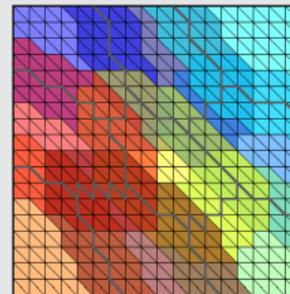
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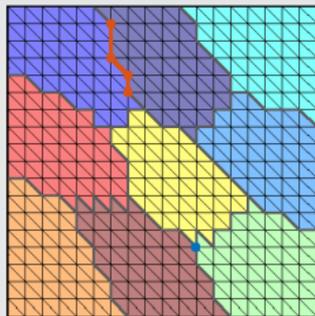


Overlap $\delta = 2h$



Coarse space

1. Interface components



Overlapping domain decomposition

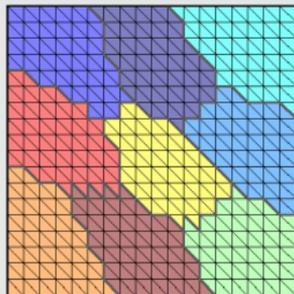
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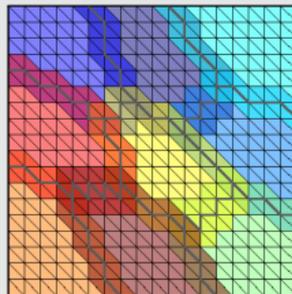
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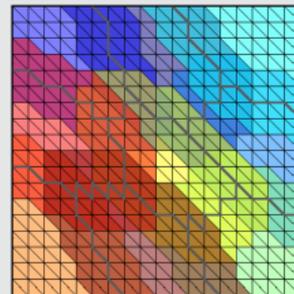
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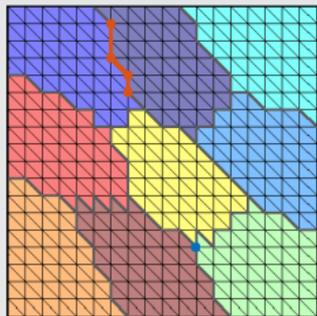


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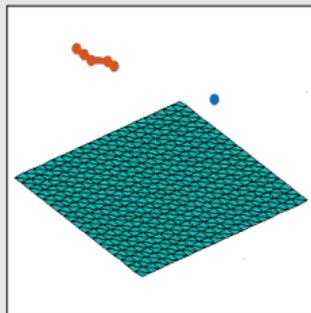


Coarse space

1. Interface components



2. Interface basis (partition of unity \times null space)



For **scalar elliptic problems**, the **null space** consists only of **constant functions**.

Overlapping domain decomposition

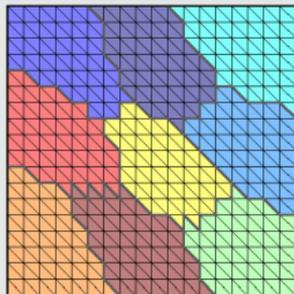
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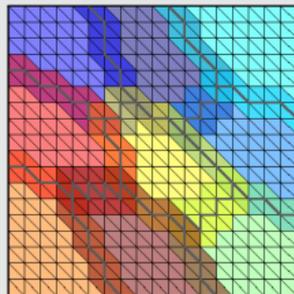
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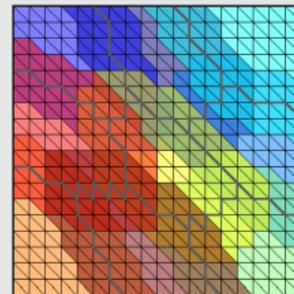
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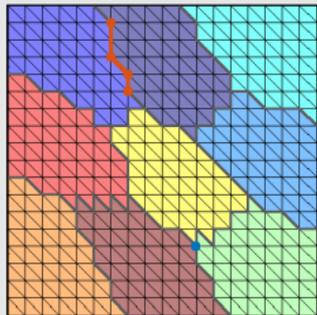


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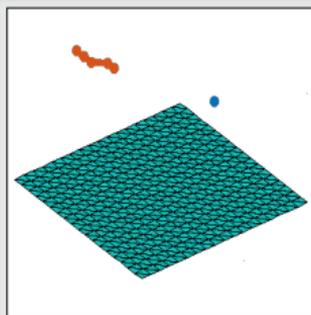


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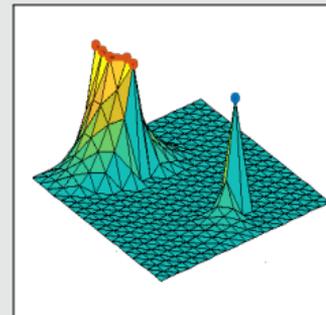


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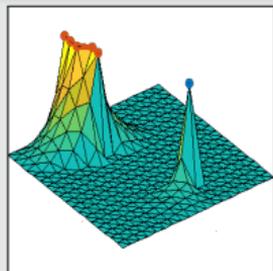
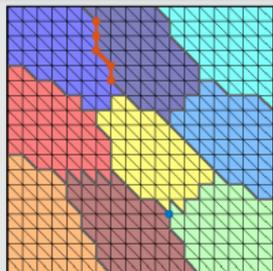
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3. Extension



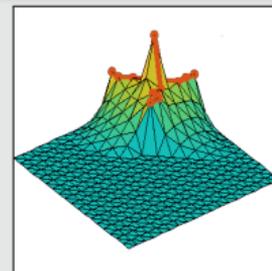
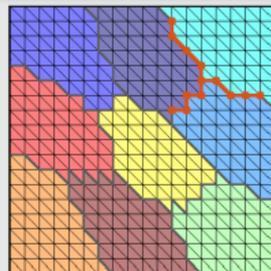
Examples of FROSch Coarse Spaces

GDSW (Generalized Dryja–Smith–Widlund)



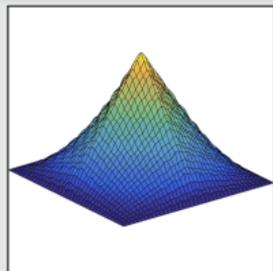
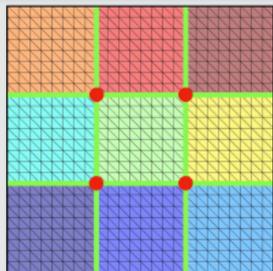
- Dohrmann, Klawonn, Widlund (2008)
- Dohrmann, Widlund (2009, 2010, 2012)

RGDSW (Reduced dimension GDSW)



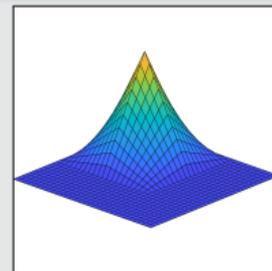
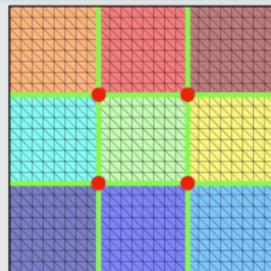
- Dohrmann, Widlund (2017)
- H., Klawonn, Knepper, Rheinbach, Widlund (2022)

MsFEM (Multiscale Finite Element Method)



- Hou (1997), Efendiev and Hou (2009)
- Buck, Iliev, and Andrä (2013)
- H., Klawonn, Knepper, Rheinbach (2018)

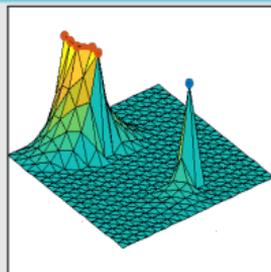
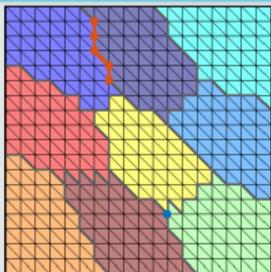
Q1 Lagrangian / piecewise bilinear



Piecewise linear interface partition of unity functions and a structured domain decomposition.

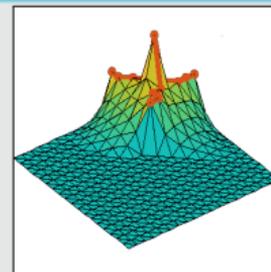
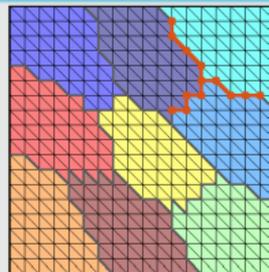
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GDSW (Generalized Dryja–Smith–Widlund)



- Dohrmann, Klawonn, Widlund (2008)
- Dohrmann, Widlund (2009, 2010, 2012)

RGDSW (Reduced dimension GDSW)



- Dohrmann, Widlund (2017)
- H., Klawonn, Knepper, Rheinbach, Widlund (2022)

For elliptic model problems, the **condition number of the (R)GDSW two-level Schwarz operator** is bounded by

$$\kappa \left(M_{(\text{R})\text{GDSW}}^{-1} \mathbf{K} \right) \leq C \left(1 + \frac{H}{\delta} \right) \left(1 + \log \left(\frac{H}{h} \right) \right)^\alpha,$$

where

- C constant (does not depend on h , H , or δ),
- H subdomain diameter,
- h element size,
- δ width of the overlap,
- $\alpha \in \{0, 1, 2\}$ power (depends on problem dimension, regularity of the subdomains, and variant of the algorithm).

Monolithic Coarse Spaces Multiphysics Problems

Monolithic (R)GDSW Preconditioners for CFD Simulations

Consider the discrete saddle point problem

$$\mathcal{A}x = \begin{bmatrix} \mathbf{K} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} = \mathbf{b}.$$

Monolithic GDSW preconditioner

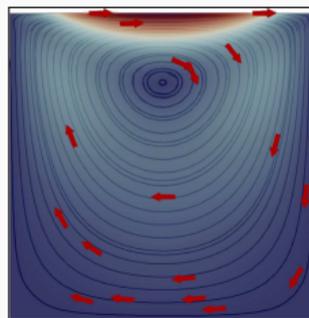
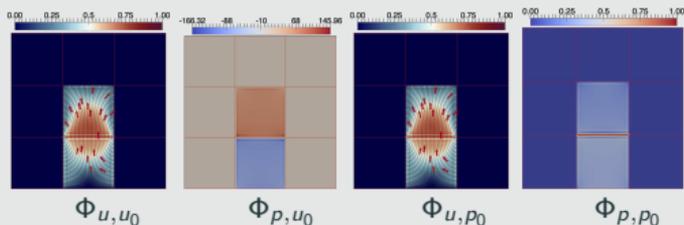
We construct a **monolithic GDSW preconditioner**

$$m_{\text{GDSW}}^{-1} = \phi \mathcal{A}_0^{-1} \phi^\top + \sum_{i=1}^N \mathcal{R}_i^\top \bar{\mathcal{P}}_i \mathcal{A}_i^{-1} \mathcal{R}_i,$$

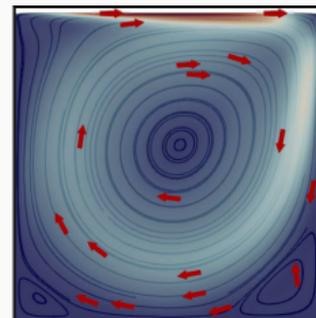
with block matrices $\mathcal{A}_0 = \phi^\top \mathcal{A} \phi$, $\mathcal{A}_i = \mathcal{R}_i \mathcal{A} \mathcal{R}_i^\top$, local pressure projections $\bar{\mathcal{P}}_i$, and

$$\mathcal{R}_i = \begin{bmatrix} \mathcal{R}_{u,i} & \mathbf{0} \\ \mathbf{0} & \mathcal{R}_{p,i} \end{bmatrix} \quad \text{and} \quad \phi = \begin{bmatrix} \Phi_{u,u_0} & \Phi_{u,p_0} \\ \Phi_{p,u_0} & \Phi_{p,p_0} \end{bmatrix}.$$

Using \mathcal{A} to compute extensions: $\phi_l = -\mathcal{A}_{ll}^{-1} \mathcal{A}_{l\Gamma} \phi_\Gamma$; cf. **Heinlein, Hochmuth, Klawonn (2019, 2020)**.



Stokes flow



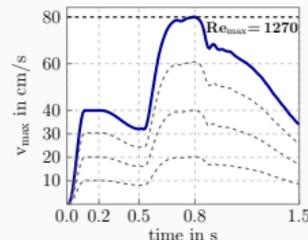
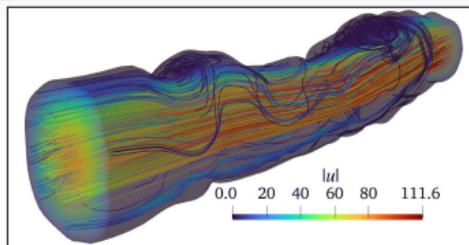
Navier–Stokes flow

Related work:

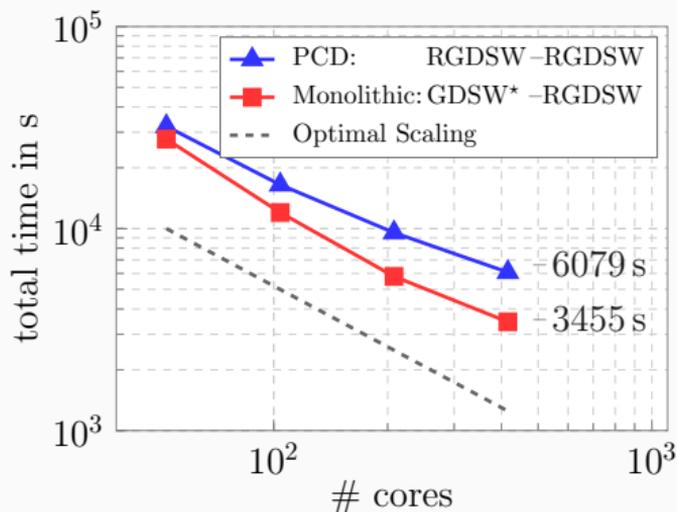
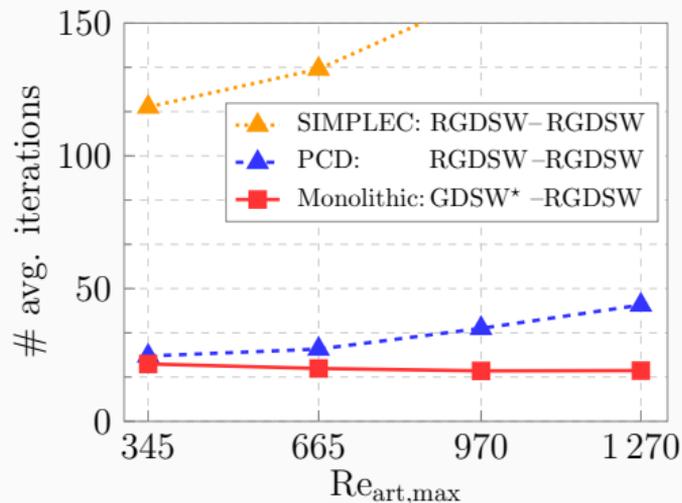
- Original work on monolithic Schwarz preconditioners: **Klawonn and Pavarino (1998, 2000)**
- Other publications on monolithic Schwarz preconditioners: e.g., **Hwang and Cai (2006)**, **Barker and Cai (2010)**, **Wu and Cai (2014)**, and the presentation **Dohrmann (2010)** at the *Workshop on Adaptive Finite Elements and Domain Decomposition Methods in Milan*.

Results for Blood Flow Simulations

- **3D unsteady flow simulation** within the geometry of a realistic artery (from [Balzani et al. \(2012\)](#)) and kinematic viscosity $\nu = 0.03 \text{ cm}^2/\text{s}$
- **Parabolic inflow profile** at inlet
- **Time discretization:** BDF-2; **space discretization:** P2-P1 elements



Cf. [Heinlein, Klawonn, Knepper, Saßmannshausen \(arXiv 2025\)](#)



More details in the talk by [Lea Saßmannshausen](#) in MS24, Thursday, 2.40pm.

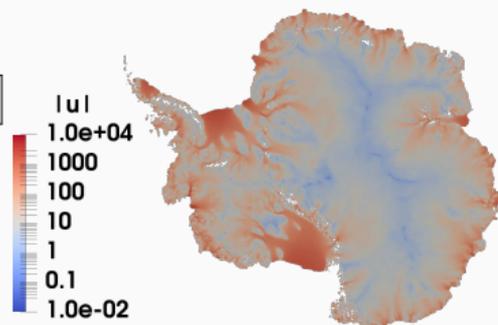


<https://github.com/SNLComputation/Albany>

The velocity of the ice sheet in Antarctica and Greenland is modeled by a **first-order-accurate Stokes approximation model**,

$$-\nabla \cdot (2\mu\dot{\epsilon}_1) + \rho g \frac{\partial s}{\partial x} = 0, \quad -\nabla \cdot (2\mu\dot{\epsilon}_2) + \rho g \frac{\partial s}{\partial y} = 0,$$

with a **nonlinear viscosity model** (Glen's law); cf., e.g., **Blatter (1995)** and **Pattyn (2003)**.



MPI ranks	Antarctica (velocity)			Greenland (multiphysics vel. & temperature)		
	4 km resolution, 20 layers, 35 m dofs			1-10 km resolution, 20 layers, 69 m dofs		
	avg. its	avg. setup	avg. solve	avg. its	avg. setup	avg. solve
512	41.9 (11)	25.10 s	12.29 s	41.3 (36)	18.78 s	4.99 s
1024	43.3 (11)	9.18 s	5.85 s	53.0 (29)	8.68 s	4.22 s
2048	41.4 (11)	4.15 s	2.63 s	62.2 (86)	4.47 s	4.23 s
4096	41.2 (11)	1.66 s	1.49 s	68.9 (40)	2.52 s	2.86 s
8192	40.2 (11)	1.26 s	1.06 s	-	-	-

Computations performed on Cori (NERSC).

Heinlein, Perego, Rajamanickam (2022)

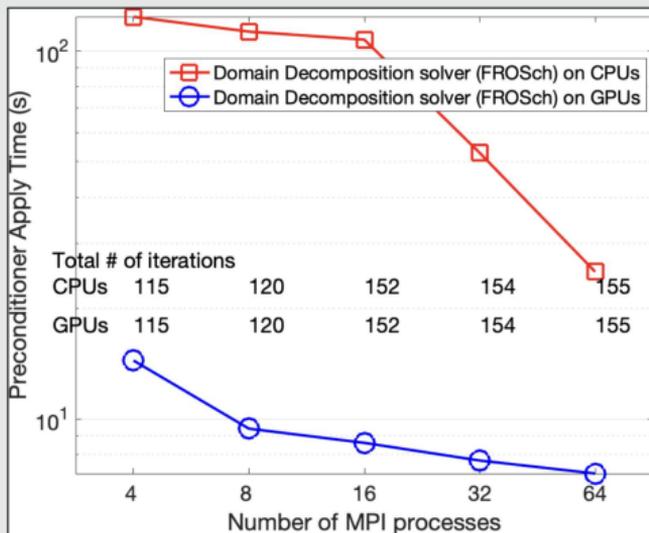
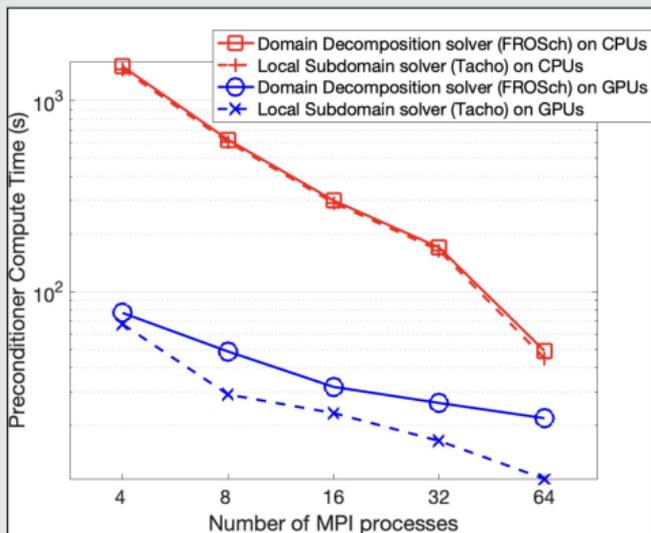
Land Ice Simulations – Fast Subdomain Solves Using Tacho

Tacho

- Multifrontal factorization with pivoting
- Impl. using KOKKOS and level-set scheduling

Cf. Kim, Edwards, Rajamanickam (2018).

Strong scaling results on a single compute node of Perlmutter (NERSC)



Cf. Yamazaki, Ellingwood, and Rajamanickam (subm. 2025).

Robust Coarse Spaces for Heterogeneous Problems

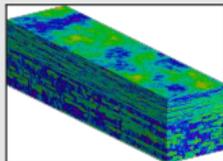
Spectral Extension-Based Coarse Spaces for Schwarz Preconditioners

Highly heterogeneous problems ...

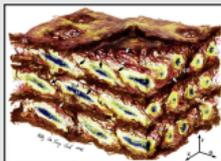
... appear in most areas of modern science and engineering:



Micro section of a dual-phase steel.
Courtesy of **J. Schröder**.



Groundwater flow (SPE10);
cf. **Christie and Blunt (2001)**.



Composition of arterial walls; taken from **O'Connell et al. (2008)**.

Spectral coarse spaces

The coarse space is **enhanced** by eigenfunctions of **local edge and face eigenvalue problems** with eigenvalues below tolerances $tol_{\mathcal{E}}$ and $tol_{\mathcal{F}}$:

$$\kappa(M_*^{-1}K) \leq C \left(1 + \frac{1}{tol_{\mathcal{E}}} + \frac{1}{tol_{\mathcal{F}}} + \frac{1}{tol_{\mathcal{E}} \cdot tol_{\mathcal{F}}} \right);$$

C does not depend on h , H , or the coefficients.

OS-ACMS & adaptive GDSW (AGDSW) (**Heinlein, Klawonn, Knepper, Rheinbach (2018, 2018, 2019)**).

Local eigenvalue problems

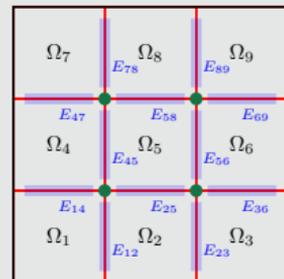
Local generalized eigenvalue problems corresponding to the edges \mathcal{E} and faces \mathcal{F} of the domain decomposition:

$$\forall E \in \mathcal{E}: \quad S_{EE} T_{*,E} = \lambda_{*,E} K_{EE} T_{*,E}, \quad \forall T_{*,E} \in V_E,$$

$$\forall F \in \mathcal{F}: \quad S_{FF} T_{*,F} = \lambda_{*,F} K_{FF} T_{*,F}, \quad \forall T_{*,F} \in V_F,$$

with **Schur complements** S_{EE} , S_{FF} with **Neumann boundary conditions** and **submatrices** K_{EE} , K_{FF} of K . We select eigenfunctions corresponding to **eigenvalues below tolerances** $tol_{\mathcal{E}}$ and $tol_{\mathcal{F}}$.

→ The corresponding coarse basis functions are **energy-minimizing extensions** into the interior of the subdomains.



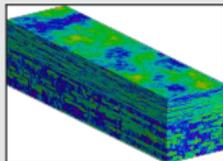
Spectral Extension-Based Coarse Spaces for Schwarz Preconditioners

Highly heterogeneous problems ...

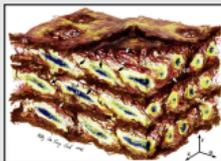
... appear in most areas of modern science and engineering:



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Spectral coarse spaces

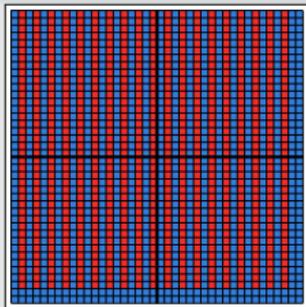
The coarse space is **enhanced** by eigenfunctions of **local edge and face eigenvalue problems** with eigenvalues below tolerances tol_g and tol_f :

$$\kappa(M_*^{-1}K) \leq C \left(1 + \frac{1}{tol_g} + \frac{1}{tol_f} + \frac{1}{tol_g \cdot tol_f} \right);$$

C does not depend on h , H , or the coefficients.

OS-ACMS & **adaptive GDSW (AGDSW)** (**Heinlein, Klawonn, Knepper, Rheinbach (2018, 2018, 2019)**).

FROSch – Channel coefficient function example



Example: 2×2 subd.'s and $H/h = 20$
Red: $\alpha = 10^6$; blue: $\alpha = 1$

- 2D Diffusion problem on unit square discretized Q1 finite elements
- $N \times N$ subdomains, $H/h = 20$, minimal algebraic overlap

# subdomains = # MPI ranks	# iterations	
	GDSW	AGDSW
2×2	105	13
4×4	502	17
8×8	1451	19
16×16	2981	19

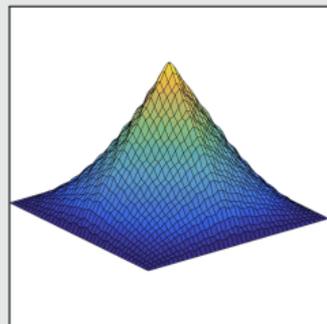
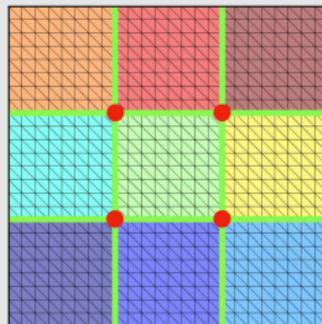
Joint work with **Axel Klawonn, Jascha Knepper, and Ichitaro Yamazaki**.

Algebraic Multiscale Coarse Space

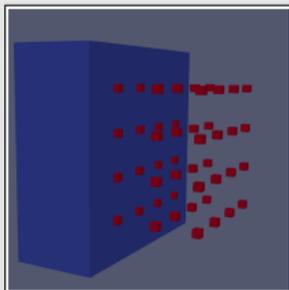
Multiscale Finite Element Method (MsFEM) (Hou and Wu, 1997)

MsFEM defines a set of coarse basis functions as the solution of the local boundary condition problem:

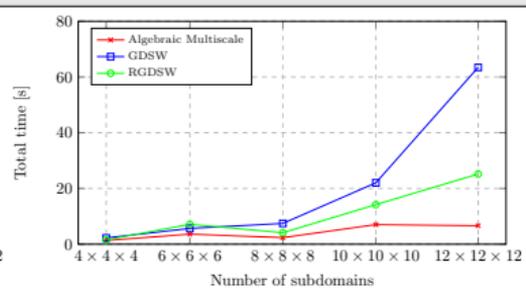
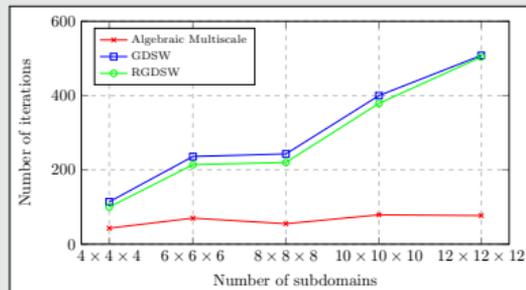
$$\begin{aligned} -\nabla \cdot (\alpha(x) \nabla \varphi_i(x)) &= 0 && \text{in } \Omega_k, \\ \varphi_i &= 0 && \text{on } \partial\Omega, \\ \nabla_{\parallel} (\alpha(x) \varphi_i(x))_{\parallel} &= 0 && \text{on } \Gamma = \partial\Omega_k \setminus \partial\Omega, \\ \varphi_i(x_j) &= \delta_{ij} && \text{for a coarse node } x_j. \end{aligned}$$



3D heterogeneous problem on DelftBlue (TU Delft)



Red: $\alpha = 10^5$; blue: $\alpha = 1$



Comparison of coarse spaces with $H/h = 16$, one layer of overlap

More details in the talk by [Filipe Cumaru](#) in MS05, Tuesday, 12.00pm.

Advances of FROSch Preconditioners for Multiphysics and Multiscale Simulations

- FROSch leverages the **Schwarz framework** and **extension-based coarse spaces** to achieve **robustness** and **scalability** while relying mostly on **algebraic information**.
- **Monolithic coarse spaces** ensure robust performance for **multiphysics problems**, e.g., strong convergence in **CFD** and scalability in **land ice** simulations.
- Robust convergence for **heterogeneous problems** requires tailored coarse spaces; recent advances include **robust multiscale** and **spectral coarse spaces** in FROSch.

Further talks on FROSch

- **Kyrill Ho** in MS27, Monday, 2.20pm (Room T23)
- **Filipe Cumarú** in MS05, Tuesday, 12.00pm (Room T04)
- **Thomas Wick** in MS06, Tuesday, at 3.00pm (Room 16B11)
- **Lea Saßmannshausen** in MS24, Thursday, 2.40pm (16B21)
- **Sebastian Kinnewig** in MS17, Thursday, at 11.40am (Room T23)

Thank you for your attention!