Project: Domain Decomposition Techniques for the Helmholtz Equation – Theoretical Investigation

Supervision: Vandana Dwarka (Numerical Analysis. EEMCS)
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Our group is world leading in the development of fast and robust solvers for discretized Helmholtz problems. The Helmholtz equation is the time-harmonic equivalent of the wave equation and is used in a wide range of engineering practices. It mainly models electromagnetic waves and is used from scattering studies in medical imaging to seismology.

Several issues arise when trying to solve the Helmholtz equation numerically. One of them is related to the efficiency of the numerical solvers. Due to the size and indefiniteness of the linear system, numerical solutions are obtained by using iterative solvers, in particular GMRES. Moreover, depending on the boundary conditions, the matrix becomes complex non-Hermitian. The challenge in designing efficient solvers is that the number of iterations to reach convergence grows with the wavenumber.

In our group, recent research has focused on developing a domain decomposition preconditioner for the Helmholtz equation, in particular a two-level Schwarz preconditioner that has wavenumber independent convergence and has reduced computational costs. One important feature here was the introduction of first-order or a higher-order Bézier interpolation to construct the coarse-spaces [1, 2]. For a simple model problem, the convergence of the iterative method can be deduced by analyzing the spectrum and eigenvalues. However, it seemed that even for the simple model, the convergence behavior could not be fully explained by the spectral analysis [3]. This MSc. project is designed to figure out why this is the case by comparing the convergence behavior and spectrum of the first-order and second-order interpolation schemes for the coarse spaces.

At the end of the project, the aim is to have developed a novel theoretical framework to understand and explain the convergence behavior of the numerical solver.The results will be crucial to the further development of scalable and accurate solvers for indefinite Helmholtz problems.

1 Research plan and schedule

In this section we outline the research plan and schedule.

- Prerequisites: real and complex analysis, linear algebra, scientific computing, familiarity with programming in Matlab or Python.
- Literature study (approx. 1 month) with topics:
  - Helmholtz equation
  - GMRES iterative method
  - Interpolation method
  - Domain decomposition method (in particular two-level Schwarz methods)
- Research phase I: Preliminary implementation for 1D/2D simple model problem using two-level Schwarz and first- and second-order interpolation schemes (approx. 2-3 months).
- Research phase II: Spectral analysis, detailed comparison of both methods including location of smallest eigenvalues and shifts (approx. 2-3 months).
- Research phase III: Finalizing thesis (approx. 1 month).

Contact

If you are interested in this project and/or have further questions please contact Vandana Dwarka, v.n.s.r.dwarka@tudelft.nl, and Alexander Heinlein, a.heinlein@tudelft.nl.

References

[1] V. Dwarka and C. Vuik. Scalable convergence using two-level deflation preconditioning for the helmholtz equation. *SIAM Journal on Scientific Computing*, 42(2):A901–A928, 2020.

[2] A. Heinlein, A. Klawonn, S. Rajamanickam, and O. Rheinbach. *FROSch: A Fast And Robust Overlapping Schwarz Domain Decomposition Preconditioner Based on Xpetra in Trilinos*, pages 176–184. Springer, 2020.

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Project: Domain Decomposition Techniques for the Helmholtz Equation – HPC Implementation

Supervision: Vandana Dwarka (Numerical Analysis. EEMCS)
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Our group is world leading in the development of fast and robust solvers for discretized Helmholtz problems. The Helmholtz equation is the time-harmonic equivalent of the wave equation and is used in a wide range of engineering practices. It mainly models electromagnetic waves and is used from scattering studies in medical imaging to seismology. The study of the Helmholtz equation and its correct simulation were and are, still to this day, a vital part of applied mathematics.

Several issues arise when trying to solve the Helmholtz equation numerically. The first issue is related to the accuracy of the numerical solution. Due to the heavy oscillatory nature of waves, the solution often contains numerical dispersion errors which translate to having phase differences. This is called the pollution error. In order to avoid this, the Helmholtz equation needs to be solved on very fine grids, leading to large linear systems. The second issue is related to the efficiency of the numerical solvers. Due to the size and indefiniteness of the linear system, numerical solutions are obtained by using iterative solvers. The challenge in designing efficient solvers is that the number of iterations to reach convergence grows with the wavenumber.

But not all hope is lost, as our group and other scientists from around the world are actively involved in finding new ways to accelerate and improve numerical solutions to the Helmholtz equation. In our group, recent progress has been made in bounding the number of iterations needed to reach convergence; from linear dependence on the wavenumber to almost wavenumber independent convergence using deflation techniques. One important feature of the deflation method is the use of higher-order parametric curves to construct the deflation vectors and coarse-spaces [4].

In this master project, the goal is to use domain decomposition techniques in order to construct an efficient and scalable solver for the Helmholtz equation, which exhibits robustness with respect to the wavenumber. Therefore, the combination of two-level overlapping Schwarz domain decomposition methods and the inclusion of these higher-order coarse-spaces from the deflation setting should be explored. On the first level of the overlapping Schwarz method, the boundary value problem on the computational domain is decomposed into smaller subproblems on overlapping subdomains. These subproblems can be solved independently, making the setup and application of the first level very efficient and easy to parallelize. However, the resulting iterative scheme might require a large number of iterations since the subproblems are only coupled weakly, that is, only through their overlap. In order to facilitate scalability with respect to the number of subdomains and robustness with respect to the wavenumber, a second level (coarse level) has to be introduced.

Here, we are specifically interested in the use of a coarse level based on extension-based coarse spaces, such as GDSW (generalized Dryja–Smith–Widlund) type coarse spaces; cf. [3]. In particular, we want to investigate the numerical and parallel scalability to large discrete Helmholtz problems. Therefore, in this project, the FROSch (Fast and Robust Overlappig Schwarz) [5] parallel domain decomposition solver package, which contains a highly-scalable parallel implementation of multilevel GDSW preconditioners, should be employed. FROSch is part of the Trilinos software library [1], which also provides all the functionality required for a parallel implementation of a discretized Helmholtz problem. FROSch allows to investigate many variants of Schwarz preconditioners, including multiplicative or additive coupling of the levels [6] and restricted additive Schwarz methods on the first level [2].

1 Research plan and schedule

In this section we outline the research plan and schedule.

- Prerequisites: linear algebra, scientific computing, parallel computing, familiarity with programming in C++ and parallel programming (distributed memory; MPI)
- Literature study (approx. 3 months) with topics:
  - Helmholtz equation
  - Iterative methods and preconditioning
  - Deflation
  - Domain decomposition methods (in particular Schwarz methods and GDSW coarse spaces)
  - Convergence analysis for deflation/DDM

- Research phase I: Design of the new preconditioner + preliminary implementation for 1D/2D problems. Incorporate into thesis. (approx. 1-2 months).
- Research phase II: Implementation in FROSch for 2D/3D and large scale numerical studies on DelftBlue supercomputer[1] at TU Delft. Incorporate into thesis. (approx 3-4 months).
- Research phase III: Finalizing thesis. (1 month).

Contact

If you are interested in this project and/or have further questions please contact Vandana Dwarka, v.n.s.r.dwarka@tudelft.nl, and Alexander Heinlein, a.heinlein@tudelft.nl.

References

[1] *The Trilinos Project Website.* `https://trilinos.github.io`.

[2] Xiao-Chuan Cai and Marcus Sarkis. A restricted additive schwarz preconditioner for general sparse linear systems. *Siam journal on scientific computing*, 21(2):792–797, 1999.

[3] C. R. Dohrmann, A. Klawonn, and O. B. Widlund. Domain decomposition for less regular subdomains: overlapping Schwarz in two dimensions. *SIAM J. Numer. Anal.*, 46(4):2153–2168, 2008.

[4] V. Dwarka and C. Vuik. Scalable convergence using two-level deflation preconditioning for the helmholtz equation. *SIAM Journal on Scientific Computing*, 42(2):A901–A928, 2020.

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[6] Andrea Toselli and Olof Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[1] https://www.tudelft.nl/dhpc/system